Smooth Anosov actions on $\mathbb{T}^3$ and other nilmanifolds by surface groups

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Abstract

Constructing Anosov actions on nilmanifolds: starting with the case of tori, we explore ways to classify Anosov automorphisms and groups acting by Anosov automorphisms. Starting with the construction of examples, we study how all of these symmetries of a given algebraic structure, a nilmanifold, can be classified.

Contents

1 Introduction

Let a *surface group* be the fundamental group of a genus g surface. Surface groups are fairly well-understood from a group theory perspective, but their dynamical properties are more mysterious. It is known that when Γ is a surface group, linear representations ρ from Γ to the general linear group of $n \times n$ real matrices $GL(n, \mathbb{R})$ are quite flexible [Bro18]. However, if we consider representations σ from Γ to the special linear group of $n \times n$ integer matrices $\mathrm{SL}(n, \mathbb{Z})$, it is not known whether such σ are flexible. One may also ask analogous questions for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, and again the representation ρ is known to be flexible while σ remains elusive. The group $\mathrm{SL}(2, \mathbb{Z})$ is natural to consider in conjunction with surface groups because all surface groups are finite index in $\mathrm{SL}(2, \mathbb{Z})$ by way of the chain of embeddings

$$\text{surface group} \to \text{von Dyck group} \to \mathrm{PSL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}).$$

These embeddings admit a natural hyperbolic geometry interpretation, which we elaborate on later in the report. Also, since $\mathrm{SL}(2, \mathbb{Z})$ embeds into $\mathrm{SL}(n, \mathbb{Z})$, we may ask flexibility questions about $\mathrm{SL}(n, \mathbb{Z})$.

The group $\mathrm{SL}(n, \mathbb{Z})$ can be considered to act on the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$ by matrix multiplication. The action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{T}^2$ is flexible, while the action of $\mathrm{SL}(3, \mathbb{Z})$ on $\mathbb{T}^3$ is rigid. We may also consider subgroups of $\mathrm{SL}(n, \mathbb{Z})$ acting on $\mathbb{T}^n$. In particular, we are interested in subgroups that contain an Anosov diffeomorphism, which is a map with nice dynamical properties, such as structural stability (a property akin to flexibility). Anosov diffeomorphisms are also interesting in that they form a class of dynamical systems that exhibit chaotic behavior, such as expansivity and sensitivity to initial conditions.

The main result in our report is as follows.

Theorem 1. *There is an explicit faithful action on the* 3*-dimensional torus* $\mathbb{T}^3$ *by surface groups containing an Anosov element.*

Given this result, it is natural to consider whether we can extend our construction to a more general class of spaces to which the 3-torus belongs. The right generalization comes from the observation that the 3-torus is the quotient of the simply-connected nilpotent Lie group $\mathbb{R}^3$ by the integer lattice $\mathbb{Z}^3$. In general, the quotient of a simply-connected nilpotent Lie group N by a lattice in N is called a *(compact) nilmanifold.* Following our construction of an embedding $\pi_1(\Sigma_2) \hookrightarrow \mathrm{SL}(3, \mathbb{Z})$, we use this construction to define a faithful action of $\pi_1(\Sigma_2)$ on the free 2-step nilpotent Lie group N on three generators, with the eventual goal of constructing an Anosov action of $\pi_1(\Sigma_2)$ on a nilmanifold with covering Lie group N.

1.1 Outline of Report

The report is organized as follows. We begin with a review of differential topology and geometry in Section 2, followed by an introduction to hyperbolic geometry and surface groups in Section 3, including the statement of Poincare's theorem and the definition of triangle groups which feature prominently in Section 6.1. We also discuss in this section a way to embed $\pi_1(\Sigma_n)$ in $\pi_1(\Sigma_2)$ for $n \geq 2$, so that constructing actions by surface groups of genus $n \geq 2$ reduces to constructing actions by the surface group of genus 2.

In Section 4, we introduce Anosov diffeomorphisms and discuss some interesting dynamical properties that they satisfy, such as C^1-structural stability. We also discuss notions of structural stability more generally for actions of groups other than $\mathbb{Z}$, and show by way of an example by Hurder in [Hur92] the existence of Anosov actions that are not structurally stable in the sense of topological deformation rigidity.

In Section 5, we provide a brief review of Lie theory with emphasis on nilpotency, then introduce the notion of nilmanifolds. Next, we construct a nilmanifold that admits an Anosov diffeomorphism, featuring the Heisenberg group. This section will serve as background for Section 6.4.

Finally, in Section 6, we state our results. First, we will prove the main result, namely Theorem 1. We do so by embedding the surface group of genus 2 into a particular triangle group, apply a result in 4, then we generalize this to any genus $g \geq 2$. Second, we see if we can construct a $\mathrm{PSL}(2, \mathbb{Z})$ action on $\mathbb{T}^3$, and determine whether it is topologically deformation rigid. Finally, we explore whether we can construct an Anosov surface group action on a non-toral nilmanifold.

1.2 Acknowledgements

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2 Differential geometry

2.1 Smooth manifolds

We now review some differential topology, mainly stating definitions as well as some facts without proof; for more details, see [Lee03], from which many of these definitions and facts have been taken either directly or with modification. Given two spaces X and Y, we denote by $X \supset\!\!\to Y$ a partial mapping from X to Y defined on an open subset of X. Given a space M, a *chart* on M is an object consisting of the following data:

1) a nonnegative integer n, called the *dimension*;
2) a map $\varphi : M \supset\!\!\to \mathbb{R}^n$, called the *coordinate map*, whose domain of definition is called the *coordinate domain*,

such that $\operatorname{im}\varphi \subseteq \mathbb{R}^n$ is open and φ is a homeomorphism from its domain of definition to its image. We denote a chart on M via the notation (n, U, φ), where n is the dimension, U is the coordinate domain, and φ is the coordinate map. Note that we can recover both n and U from φ itself, with n being the dimension of the codomain of φ and U the domain of definition of φ; thus, we often omit one or both pieces of information when naming a chart.

Fix $r \in \mathbb{N} \cup \{\infty\}$. Two charts φ and ψ on M are said to be C^r*-compatible* (or *smoothly compatible* when $r = \infty$) if $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are C^r in the usual sense. An *atlas* on M is an object consisting of the following data:

1) a nonnegative integer n, called the *dimension*;
2) a set $\mathcal{A}$ of n-dimensional charts on M,

such that the coordinate domains of the charts in $\mathcal{A}$ cover M. The atlas is said to be C^r (or *smooth* when $r = \infty$) if the charts in $\mathcal{A}$ are pairwise C^r-compatible. A C^r*-manifold* is an object consisting of the following data:

1) a second countable Hausdorff space M.
2) a nonnegative integer $n = \dim M$, called the *dimension*.
3) a maximal n-dimensional C^r-atlas on M called the C^r*-structure* of M (or *smooth structure* when $r = \infty$), whose members are called the C^r*-charts* of M (or *smooth charts* when $r = \infty$).

A C^∞-manifold is more often called a *smooth manifold*—from here on out, we restrict our attention to smooth manifolds.

Given a topological space M and an n-dimensional smooth atlas $\mathcal{A}$ on M, there is a unique extension of $\mathcal{A}$ to a maximal n-dimensional smooth atlas $\overline{\mathcal{A}}$ on M, called the *smooth structure on M determined by $\mathcal{A}$*—this is just the collection of all n-dimensional charts on M that are smoothly compatible with every chart in $\mathcal{A}$. This useful fact allows us to define smooth structures on topological spaces without having to specify every chart; for instance, the identity map on $\mathbb{R}^n$ constitutes a one-element smooth atlas on $\mathbb{R}^n$ and thereby determines a smooth structure on $\mathbb{R}^n$, called the *standard smooth structure* on $\mathbb{R}^n$. If M is a smooth manifold, an *open submanifold* of M is an open subset $U \subseteq M$ endowed with the smooth structure on U determined by the set of all smooth charts of M whose coordinate domains are contained in U.

For each $r \in \mathbb{N} \cup \{\infty\}$, a set-map $f : M \to N$ between smooth manifolds is said to be C^r (or *smooth* when $r = \infty$) if, roughly speaking, f is locally C^r. More precisely, we require that for each $p \in M$, there exist smooth charts (U, φ) and (V, ψ) at p and $f(p)$, respectively, so that $f(U) \subseteq V$ and $\psi \circ f \circ \varphi^{-1}$ (called a *coordinate representative* of f) is C^r in the usual sense. The identity maps are C^r, and the composite of any two C^r maps is C^r. That is to say, smooth manifolds together with C^r maps form a *category*, called the *C^r-category of smooth manifolds*. The isomorphisms in this category (i.e., the C^r maps with C^r inverse) are called *C^r diffeomorphisms*. Note that every C^k map is C^r whenever $0 \le r \le k \le \infty$ and that the C^0 maps are precisely the continuous maps. In particular, C^r diffeomorphisms are also homeomorphisms.

If M and N are smooth manifolds, we denote by $\mathrm{Diff}^r(M, N)$ the set of all C^r diffeomorphisms from M to N. In general, a C^r diffeomorphism cannot be interpreted as a relabelling of smooth manifolds; however, this interpretation becomes valid in the case $r = \infty$.

Suppose M is an n-dimensional smooth manifold. Recall that to each point $p \in M$, we associate an n-dimensional real vector space T_pM, called the *tangent space* to M at p. Intuitively, the tangent space to M at a point can be thought of as the best linear approximation of M near that point. Furthermore, to each smooth map $f : M \to N$ and point $p \in M$, we associate a real linear map $df_p : T_pM \to T_{f(p)}N$, called the *differential* of f at p—this notion generalizes the notion of derivatives of smooth maps between Euclidean spaces. The differential operator preserves function composition and identity maps; in the category-theoretic lingo, we say that it defines a functor from the category of pointed smooth manifolds (with pointed smooth maps as morphisms) to the category of real vector spaces.

2.2 Riemannian manifolds

A *Riemannian metric* on M is a map that associates to each point $p \in M$ a real inner product $\langle \cdot, \cdot \rangle_p$ on T_pM. We require that this association be 'smooth,' which should, for now, be thought of as a technical condition. A *Riemannian manifold* is a smooth manifold equipped with a Riemannian metric. An *isometry* between two Riemannian manifolds is an inner product-preserving diffeomorphism, or, in other words, a relabelling of the underlying sets that preserves the Riemannian manifold structure. If M is a Riemannian manifold, we denote by $\mathrm{Isom}(M)$ the set of all self-isometries of M.

Every connected Riemannian manifold M comes equipped with a canonical distance function that turns it into a metric space. Given two points $p, q \in M$, a *smooth curve segment* in M from p to q is a smooth map $\gamma : [a, b] \to M$ for some real numbers $a < b$, where $\gamma(a) = p$ and $\gamma(b) = q$. We define the *length* of

γ by the formula

$$\operatorname{length}\gamma = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\,dt.$$

The *distance* $d(p, q)$ between two points p and q in M is then defined to be the greatest lower bound of all lengths of smooth curve segments from p to q.

We conclude this section with a discussion of geodesics, which is often discussed in the context of Riemannian geometry but may be defined in a more general setting. Suppose M is a metric space. Loosely speaking, a *geodesic* in M is a locally distance-minimizing curve in M. Formally, we define a geodesic in M to be a local isometry (in the usual sense for metric spaces) from an interval $I \subseteq \mathbb{R}$ into M; that is, a map $f : I \to M$ such that every point $t \in I$ has a neighborhood $U \subseteq I$ for which $f|_U : U \to M$ is an isometry. A metric space isometry from a real interval into M is called a *global geodesic*. For instance, the geodesics in Euclidean spaces $\mathbb{R}^n$ are straight lines, in accordance with our intuition.

3 Hyperbolic geometry

3.1 The hyperbolic plane

In this section, we introduce some of the basic objects and results of hyperbolic geometry, following the notes of Walkden [Wal19]. The *hyperbolic plane* is the open submanifold $\mathbb{H}^2 = \{x + iy : x \in \mathbb{R} \text{ and } y > 0\}$ of $\mathbb{C}$ equipped with the following Riemannian metric: for each $z \in \mathbb{H}^2$ and $v, w \in T_z\mathbb{H}^2 \simeq \mathbb{R}^2$, we define

$$\langle v, w\rangle_z = \frac{v \cdot w}{(\operatorname{Im} z)^2}.$$

Alternatively, we can endow the open submanifold $\mathbb{B}^2 = \{z \in \mathbb{C} : |z| < 1\}$ of $\mathbb{C}$ with a Riemannian manifold structure by defining

$$\langle v, w\rangle_z = \frac{4\, v \cdot w}{\left(1 - |z|^2\right)^2}$$

for each $z \in \mathbb{B}^2$ and $v, w \in T_z\mathbb{B}^2 \simeq \mathbb{R}^2$. These two Riemannian manifolds are isometric via the isometry $f : \mathbb{H}^2 \to \mathbb{B}^2$ given by the rule

$$f(z) = \frac{z - i}{iz - 1}.$$

Thus, as far as the Riemannian manifold structure is concerned, it makes no difference whether we work in $\mathbb{H}^2$ or $\mathbb{B}^2$. Note that the inner products on the tangent spaces to $\mathbb{H}^2$ and $\mathbb{B}^2$ are positive scalar multiples of the usual dot product, and so the notion of angles in these spaces coincides with the notion of angles in $\mathbb{C} = \mathbb{R}^2$.

Suppose $\gamma : [a, b] \to \mathbb{H}^2$ is a smooth curve segment. By definition, the length of γ can be computed as follows:

$$\begin{aligned} \operatorname{length} \gamma &= \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t)\rangle_{\gamma(t)}}\, dt \\ &= \int_a^b \sqrt{\frac{\gamma'(t) \cdot \gamma'(t)}{(\operatorname{Im} \gamma(t))^2}}\, dt = \int_a^b \frac{|\gamma'(t)|}{\operatorname{Im} \gamma(t)}\, dt. \end{aligned}$$

If γ were a smooth curve segment in $\mathbb{B}^2$, the length of γ can be computed similarly, yielding the formula

$$\operatorname{length} \gamma = \int_a^b \frac{2\,|\gamma'(t)|}{1 - |\gamma(t)|^2}\, dt.$$

Recall that the notion of lengths of curves in the hyperbolic plane is given to us from its Riemannian manifold structure. It turns out that the Riemannian manifold structure also gives us a notion of area. While we choose not to define this formally, this notion of area looks as follows in the hyperbolic plane: given a sufficiently "nice" subset $S \subseteq \mathbb{H}^2$ (including, for instance, the open and closed subsets), the area of S is given by the formula

$$\operatorname{area} S = \iint_S \frac{1}{y^2}\, dx\, dy.$$

Similarly, a sufficiently nice subset $S \subseteq \mathbb{B}^2$ has area

$$\operatorname{area} S = \iint_S \frac{4}{\left(1-(x^2+y^2)\right)^2}\, dx\, dy.$$

3.2 Mobius transformations

Suppose a, b, c, and d are real numbers with $ad-bc>0$. These numbers induce an (orientation-preserving) isometry $\gamma : \mathbb{H}^2 \to \mathbb{H}^2$ given by the rule

$$\gamma(z) = \frac{az+b}{cz+d}.$$

This map is called the *Mobius transformation* of $\mathbb{H}^2$ induced by a, b, c, d. Furthermore, the association $\mathrm{GL}^+(2,\mathbb{R}) \to \mathrm{Isom}^+(\mathbb{H}^2)$ from the group of real 2×2 matrices with positive determinant to the group of orientation-preserving self-isometries of $\mathbb{H}^2$ given by the definition above is a surjective group homomorphism with kernel $\{\lambda I : \lambda \in \mathbb{R} \setminus \{0\}\}$. Via the isometry $\mathbb{H}^2 \simeq \mathbb{B}^2$, we may similarly define Mobius transformations of $\mathbb{B}^2$. A Mobius transformation is determined completely by its values at three distinct points.

3.3 Hyperbolic polygons

In either $\mathbb{H}^2$ or $\mathbb{B}^2$, the geodesic images are the segments of circles and lines that intersect the boundary of the space (regarded as a subset of $\mathbb{C}$) at right angles. Given any two points z and w in the hyperbolic plane or its boundary, there is one and only one geodesic that joins z with w—we denote this geodesic by $[z, w]$.

Given a list $z_1, \ldots, z_n$ of points in the hyperbolic plane or its boundary, the *hyperbolic polygon* generated by $z_1, \ldots, z_n$ is defined to be the open region in the hyperbolic plane bounded by the geodesics $[z_1, z_2]$, $[z_2, z_3]$, $\ldots, [z_{n-1}, z_n]$, $[z_n, z_1]$ (called the *geodesic sides* of the hyperbolic polygon corresponding to $z_1, \ldots, z_n$). We also call such an object a *hyperbolic n-gon*. Note that any hyperbolic n-gon with generating list of vertices $z_1, \ldots, z_n$ is also a hyperbolic $(n+1)$-gon, generated by the vertices $z_1, \ldots, z_{n+1}$ with z_{n+1} being any point on the geodesic segment joining z_n to z_1.

We now state a theorem that relates the area of a hyperbolic polygon with its internal angles.

Theorem 2 (Gauss-Bonnet for hyperbolic polygons)**.** *Suppose that P is a hyperbolic polygon with generating list of vertices $z_1, \ldots, z_n$. Then*

$$\operatorname{area} P = (n-2)\pi - (\angle z_1 + \ldots + \angle z_n),$$

where $\angle z_j$ denotes the internal angle of P at z_j.

In particular, Theorem 2 says that any hyperbolic polygon with generating vertices $z_1, \ldots, z_n$ must satisfy $\angle z_1 + \ldots + \angle z_n < (n-2)\pi$. The converse is also true: there exists a hyperbolic polygon with internal angles $\theta_1, \ldots, \theta_n$ whenever $\theta_1, \ldots, \theta_n$ are positive real numbers satisfying $\theta_1 + \ldots + \theta_n < (n-2)\pi$. For instance, given any three nonnegative real numbers θ, φ, ψ for which $\theta + \varphi + \psi < \pi$, there exists a hyperbolic triangle with internal angles θ, φ, ψ.

3.4 Poincare's theorem

Let P be a hyperbolic polygon endowed with a fixed choice of a generating list of vertices. Given two (possibly equal) geodesic sides s and t, a *side-pairing transformation that pairs s with t* is a Mobius transformation that maps s onto t and takes P into its complement in the hyperbolic plane. Suppose we have a partition of the set of geodesic sides into pairs (i.e., two-element sets), as well as, for each pair $\{s, t\}$ in the partition, a side-pairing transformation $\gamma_{\{s,t\}}$ that pairs either of s and t with the other (so that $\gamma_{\{s,t\}}^{-1}$ pairs either t or s, respectively, with the other).

We now describe a procedure that will generate what are called the *elliptic cycles* of the above setup. Given a pair s, t of adjacent geodesic sides with vertex v between them, we define $*(v, s) = (v, t)$. For any vertex-side pair (v, s), we also denote by $\gamma(v, s)$ the vertex-side pair to which (v, s) is mapped upon applying the side-pairing transformation associated with s.

Begin by drawing an edgeless graph whose vertices coincide with our initial choice of vertices of P. Fix a vertex-side pair (v_0, s_0). Now apply γ and $*$ repeatedly until the pair (v_0, s_0) recurs. During each $\gamma \to *$ iteration, draw an (undirected) edge between the vertex right before the iteration and the vertex right after. If there remains a vertex in the graph with degree zero, choose such a vertex and incident geodesic side and repeat the above process upon replacing (v_0, s_0) with this new vertex-side pair.

Once this procedure terminates, the resulting graph will be a disjoint union of cycles (including those of length one). The components of this graph are called the *elliptic cycles* of our system (which consists of the polygon and our choice of vertices, geodesic sides, and side-pairing transformations). For each elliptic cycle C with vertices $v_1, \ldots, v_k$, the *angle sum* of C, denoted by $\mathrm{sum}(C)$, is the sum of the internal angles of P at $v_1, \ldots, v_k$. We say that C satisfies the *elliptic cycle condition* if there exists a positive integer m, called the *elliptic cycle number*, for which $m \cdot \mathrm{sum}(C) = 2\pi$. We are now ready to state Poincare's theorem for hyperbolic polygons.

Theorem 3 (Poincare)**.** *Let P be a hyperbolic polygon equipped with a fixed choice of vertices, geodesic sides, and side-pairing transformations $\alpha_1, \ldots, \alpha_n$ as above. Let $C_1, \ldots, C_\ell$ be the associated elliptic cycles. Suppose that $C_1, \ldots, C_\ell$ satisfy the elliptic cycle condition with elliptic cycle numbers $m_1, \ldots, m_r$. Then $\langle \alpha_1, \ldots, \alpha_n \rangle \subseteq \mathrm{Isom}^+(\mathbb{H}^2)$ has presentation $\langle \alpha_1, \ldots, \alpha_n \mid \gamma_1^{m_1}, \ldots, \gamma_\ell^{m_\ell} \rangle$, where*

γ_j is a word in $\alpha_1, \ldots, \alpha_n$ obtained by applying the procedure described above, starting at a vertex-side pair with vertex in C_j and recording, from right to left, the side-pairing map (or its inverse) used at each $\gamma \to *$ iteration.

We use this Theorem in Section 6.1 in order to embed $\pi_1(\Sigma_2)$ in the triangle group $D(3,3,4)$, which we now describe.

3.5 Triangle groups

Suppose that $p, q, r \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. The *triangle group* $D(p,q,r) := \langle a, b | a^p = b^q = (ab)^r \rangle$ is the group of orientation-preserving isometries of the elliptic, Euclidean, or hyperbolic plane that preserve a given tiling of the plane by triangles with internal angles π/p, π/q, and π/r. Consider one such triangle with vertices u, v, and w and internal angles π/p, π/q, and π/r, respectively. The generator a of $D(p,q,r)$ rotates $2\pi/p$ clockwise about u, and the generator b rotates $2\pi/q$ clockwise about v.

Remark 1. The group $D(p,q,r)$, which we have called a triangle group, is known, in other contexts, as a *von Dyck group*. In such contexts, the corresponding *triangle group*, often denoted by $\Delta(p,q,r)$, is the group of (not necessarily orientation-preserving) isometries of the plane that preserve a given tiling of the plane by triangles with internal angles π/p, π/q, and π/r. The von Dyck group $D(p,q,r)$ lives inside $\Delta(p,q,r)$ as an index-two subgroup. From here on out, we will refer to $D(p,q,r)$ by the name *triangle group*.

Remark 2. With one caveat in Remark 5, we will consider only $D(p,q,r)$ for which
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$
These such $D(p,q,r)$ can be considered as the group of orientation-preserving isometries of the hyperbolic plane preserving the triangular tiling.

3.6 Surface groups

Definition 1. Let X be a topological space. The *fundamental group* of X at base-point $x \in X$, written $\pi_1(X, x)$, is the set
$$\{\text{loops based at } x\}/\text{homotopy}$$
with the operation of concatenation of loops.

If X is path-connected, then the isomorphism class of $\pi_1(X, x)$ is independent of the choice of the base point x. From now on, all topological spaces we work with will be path-connected, so we will just write $\pi_1(X)$.

Definition 2. If X is path-connected and has trivial $\pi_1(X)$, then we say X is *simply connected*.

Figure 1: If you identify pairs of edges as indicated by the colors, you obtain the 2-holed torus. The edges are labeled $0, 1, ..., 7$ starting at the top edge and proceeding counterclockwise around the octagon.

There is a classification theorem that states that every connected closed orientable surface is homeomorphic to a genus $n \geq 0$ surface. Such a surface is called an *n-holed torus*, and we denote it by Σ_n.

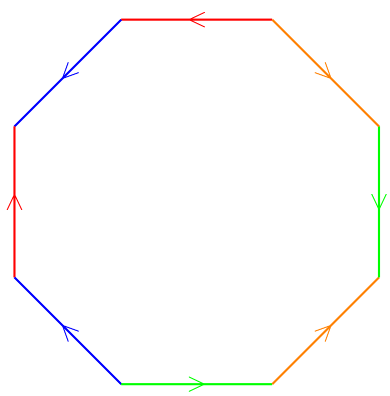
Definition 3. A *surface group* is a group isomorphic to $\pi_1(\Sigma_n)$ for some $n \geq 1$.

If you draw out a $4n$-gon, orient the sides appropriately, and then identify pairs of edges, you get an n-holed torus [Hat01]. More precisely, we number the sides from 0 to $4n - 1$ while going around the polygon counterclockwise), and then choose an orientation for each side based on the following rule. An edge labeled with the value E is oriented

$$\begin{array}{rl} \text{clockwise} & \text{if } E \equiv 0 \text{ or } 1 \pmod 4 \\ \text{counterclockwise} & \text{if } E \equiv 2 \text{ or } 3 \pmod 4 \end{array}$$

To get the edge identifications, we identify an edge E that is $0 \pmod 4$ with the edge $E' = E + 2$, and identify an edge E that is $1 \pmod 4$ with the edge $E' = E + 2$.

Here is an illustration of the fundamental polygon with edge identifications for the 2-holed torus:



This gives rise to the following presentation of the fundamental group:

$$\langle a_1, b_1, a_2, b_2, ..., a_n, b_n \mid [a_1, b_1][a_2, b_2]...[a_n, b_n]\rangle.$$

3.7 Fundamental groups of orbifolds

The precise definition of an orbifold is not so important, but we reproduce the definition from [Thu97] for completeness.

Definition 4. An *orbifold* O is a Hausdorff space X_O with additional structure. The space X_O can be covered by a finite collection of open sets U_i, and this collection is closed under finite intersections. Each U_i is associated with

1. a finite group Γ_i,
2. an action of Γ_i on an open $\tilde{U}_i \subset \mathbb{R}^n$,
3. a homeomorphism $\phi_i : U_i \to \tilde{U}_i/\Gamma_i$.

Whenever $U_i \subset U_j$ there is an injective homomorphism

$$f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$$

and an embedding

$$\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$$

equivariant with respect to f_{ij} so that this diagram commutes:

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\phi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\phi_{ij}"] & \tilde{U}_j/\Gamma_j \arrow[d, "f_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\phi_i"] \arrow[r, hook] & U_j \arrow[u, "\phi_j"']
\end{tikzcd}$$

Remark 3. Orbifolds are a generalization of manifolds: orbifolds are locally modeled as open subsets U_i of $\mathbb{R}^n$ quotiented by finite group actions, and for manifolds these groups are trivial.

Remark 4. An orbifold that is locally modeled as open subsets U_i of $\mathbb{R}^n$ is called an *n-orbifold.*

The main orbifolds we will consider in this report are $S^2(p, q, r)$ which is a sphere with cone points of orders p, q, r.

Definition 5. At a *cone point* of order p, we have $\mathbb{Z}/p\mathbb{Z}$ acting on $\mathbb{R}^2$ by rotations by $2\pi/p$ around a point.

3.8 Injection of surface groups into triangle group

We show the existence of injective maps $\alpha : \pi_1\Sigma_2 \to D(3, 3, 4)$ and $\beta_n : \pi_1\Sigma_n \to \pi_1\Sigma_2$, so that $\alpha \circ \beta_n : \pi_1\Sigma_n \to D(3, 3, 4)$ gives an injection of the surface group of an n-holed torus into $D(3, 3, 4)$.

First, we handle α. In 2011, Long, Reid, and Thistlethwaite gave a family of faithful integral 3-dimensional representations of $D(3, 3, 4)$ [LRT11]. More precisely,

Theorem 4. *Consider a representation of the von Dyck group*

$$D(3,3,4) = \langle a, b \mid a^3 = b^3 = (ab)^4 = 1 \rangle$$

given by

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$b = \begin{pmatrix} 1 & 2 - t + t^2 & 3 + t^2 \\ 0 & -2 + 2t - t^2 & -1 + t - t^2 \\ 0 & 3 - 3t + t^2 & (-1+t)^2 \end{pmatrix}$$

When $t \in \mathbb{R}$ this representation is

1. *discrete*
2. *faithful.*

When $t \in \mathbb{Z}$ the image of the representation is

1. *Zariski dense in $SL(3,\mathbb{R})$,*
2. *infinite index in $SL(3,\mathbb{Z})$*
3. *freely indecomposable,*
4. *purely semisimple,*
5. *word hyperbolic,*
6. *and has Property FA.*

3.8.1 Injection of $\pi_1(\Sigma_n)$ into $\pi_1(\Sigma_2)$

Let $\pi_1(\Sigma_2)$ have presentation $\langle a, b, c, d \mid [a,b][c,d] \rangle$.

Theorem 5. *There is a presentation of $\pi_1(\Sigma_n)$ in $\pi_1(\Sigma_2)$ given by*

$$\pi_1(\Sigma_n) \cong \langle a^{n-1}, b, c, d, aca^{-1}, ada^{-1}, ..., a^{n-2}ca^{-(n-2)}, a^{n-2}da^{-(n-2)} \rangle.$$

Proof. We show how to find a presentation of $\pi_1(\Sigma_3)$ in $\pi_1(\Sigma_2)$, and then explain how this generalizes to higher genus surface groups.

Choose the basepoint for Σ_3 as in figure 2. The generators of $\pi_1(\Sigma_3)$ consist of three loops A_1, B_1, and C_1 going around each of the holes in Σ_3, and three loops A_2, B_2, and C_2 going through each of the holes in Σ_3. We similarly choose a basepoint for Σ_2 and label the generators a, b, c, and d for $\pi_1(\Sigma_2)$ as in figure 2.

The surface group of genus 3 has 180 degree rotational symmetry, and the quotient map which identifies points under this rotation is a two fold covering $q\colon \Sigma_3 \to \Sigma_2$ of Σ_2 by Σ_3. Therefore, it induces an embedding of fundamental

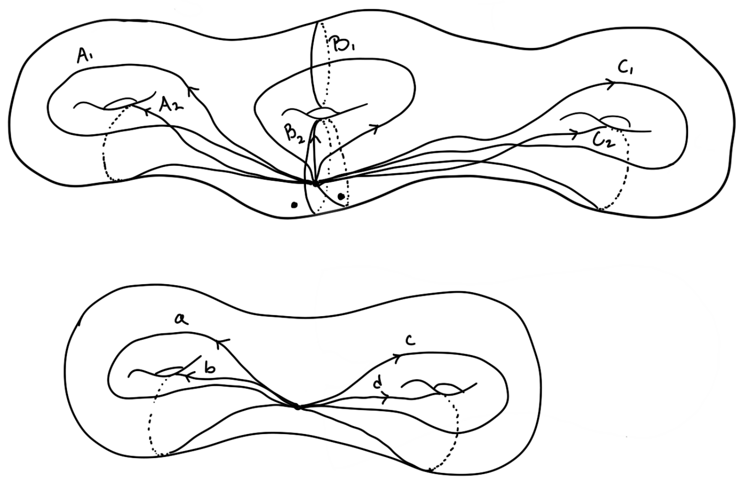


Figure 2: Generators of $\pi_1(\Sigma_3)$ and $\pi_1(\Sigma_2)$

groups $\pi_1(\Sigma_3) \hookrightarrow \pi_1(\Sigma_2)$, so to find a presentation of $\pi_1(\Sigma_3)$ in $\pi_1(\Sigma_2)$ it is enough to find where the generators of $\pi_1(\Sigma_3)$ are sent to. We can think of this as cutting Σ_3 in half through the middle hole, and pasting the two ends of each half together. The left half will need to be rotated so that this pasting occurs on the left. See figure 3.

The generators B_2, C_1, and C_2 are not affected by this pasting, so they are just sent to b, c and d. The generator B_1 is wrapped around the left hole of Σ_2 twice (once for each half of Σ_3), so it is sent to a^2. The generators A_1 and A_2 are both conjugated by a, since the action of gluing the two legs of the left half of Σ_3 stretches them around the left hole of Σ_2.

Thus, this covering $\Sigma_3 \to \Sigma_2$ gives us a map $\pi_1(\Sigma_3) \hookrightarrow \pi_1(\Sigma_2)$ which sends the generators

$$\begin{array}{lll} A_1 \mapsto aca^{-1} & B_1 \mapsto a^2 & C_1 \mapsto c \\ A_2 \mapsto ada^{-1} & B_2 \mapsto b & C_2 \mapsto d. \end{array}$$

When we add an extra handle to Σ_3 to obtain Σ_4, there are two new generating loops D_1, D_2 for the fundamental group. They will be sent to the loops a^2ca^{-2} and a^2da^{-2} under the covering map $\Sigma_4 \to \Sigma_2$, since they wrap around the left hole of Σ_2 once each time we glue together the $4-1-1=2$ handles of Σ_4 which the loops D_1 and D_2 pass through.

Continuing in this manner, we see that we obtain the stated presentation for $\pi_1(\Sigma_n)$ in $\pi_1(\Sigma_2)$.

$\square$

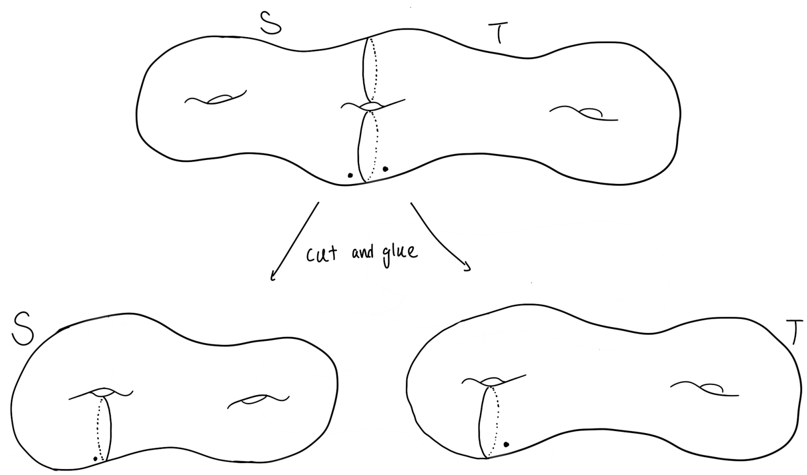


Figure 3: Covering map $\Sigma_3 \to \Sigma_2$

3.9 $\mathbb{Z}^2$ is not a subgroup of $D(p, q, r)$

Theorem 6. *$\mathbb{Z}^2$ is not a subgroup of $D(p, q, r) = \langle x, y, z \mid x^p = y^q = z^r = xyz = 1\rangle$ with $1/p + 1/q + 1/r < 1$.*

This problem can be stated entirely in terms of group theory, without defining the notion of surface group, and can be solved without using surface groups. However, there is a very clean proof using them that we give here.

First, we recognize $\mathbb{Z}^2 \cong \pi_1(\Sigma_1)$, and the only orbifold up to isomorphism that has fundamental group isomorphic to $\mathbb{Z}^2$ is Σ_1. By the Galois correspondence for covering spaces, an injective map $\pi_1(\Sigma_1) \hookrightarrow \pi_1(S^2(p, q, r))$ corresponds to a covering map $S^2(p, q, r) \to \Sigma_1$. Since $S^2(p, q, r)$ is compact, the covering map cannot be infinite sheeted, so say that it is a k-sheeted cover for some positive integer k. Then $\chi(S^2(p, q, r)) = k \cdot \chi(\Sigma_1)$, where χ denotes the Euler characteristic. Note that $\chi(\Sigma_1) = 2 - 2 \cdot 1 = 0$, so this equation implies $\chi(S^2(p, q, r)) = 0$. However, according to a formula in [Thu97], it is actually the case that $\chi(S^2(p, q, r)) < 0$, contradiction. Thus there is no injection $\mathbb{Z}^2 \hookrightarrow D(p, q, r)$, as desired.

Remark 5. If we relax the hyperbolic condition on $D(p, q, r)$, then we might be able to find a copy of $\mathbb{Z}^2$. For example, consider $\langle xz^2, yx^2\rangle < D(3, 3, 3)$. Considering the tiling of the Euclidean plane by equilateral triangles, we observe that xz^2 and yx^2 correspond to translations in linearly independent directions, so they are infinite order and commute with one another.

4 Dynamical systems

In this section, we start by examining "hyperbolic toral automorphisms," which are linear maps on tori with no eigenvalues of modulus one. We then generalize this to the notion of Anosov diffeomorphisms on compact smooth manifolds, and discuss some of their properties. In particular, we sketch the proof that Anosov diffeomorphisms on compact smooth manifolds are *structurally stable*, i.e. that any small C^1-perturbation of an Anosov diffeomorphism f remains conjugate to f via a homeomorphism.

We conclude this section by discussing the dynamics of group actions. We give a proof that the standard algebraic action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$ is not topologically rigid, following the example of Hurder in [Hur92].

4.1 Hyperbolic Toral Automorphisms

A matrix $A \in \mathrm{SL}(n,\mathbb{Z})$ preserves the integer lattice $\mathbb{Z}^n < \mathbb{R}^n$; hence, the linear map $x \mapsto Ax$ on $\mathbb{R}^n$ induces a map f_A on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. If A has no eigenvalues of modulus one, then we call A a *hyperbolic* matrix, and we call the map $f_A\colon \mathbb{T}^n \to \mathbb{T}^n$ a *hyperbolic toral automorphism.*

4.1.1 Example: The cat map

Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The hyperbolic toral automorphism $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ induced by this matrix is called the *(Arnold's) cat map.* The eigenvalues of A are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \lambda_1^{-1} = \frac{3-\sqrt{5}}{2},$$

with corresponding eigenvectors

$$v_1 = \left(\frac{1+\sqrt{5}}{2}, 1\right) \quad \text{and} \quad v_2 = \left(\frac{1-\sqrt{5}}{2}, 1\right).$$

Notice that f_A exponentially expands vectors parallel to v_1, and exponentially contracts vectors parallel to v_2. Thus, the basis $\{v_1, v_2\}$ gives us a splitting $\mathbb{R}^2 \cong \mathbb{R}v_1 \oplus \mathbb{R}v_2$ of $\mathbb{R}^2$ into an expanding subbundle $\mathbb{R}v_1$ and a contracting subbundle $\mathbb{R}v_2$. Moreover, these subbundles are invariant under the action of A on $\mathbb{R}^2$.

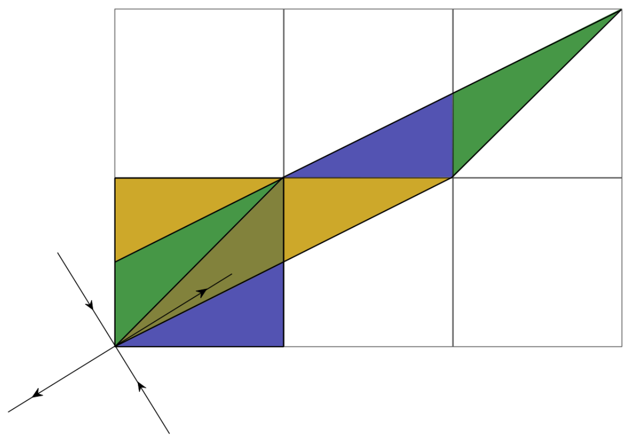


Figure 4: The action of the cat map f_A on the unit square, from Wikipedia [Com08]. The axes in the bottom left represent the expanding and contracting eigenvectors of A, and the coloring shows where the pieces of the stretched square end up after we quotient by $\mathbb{Z}^2$.

4.1.2 Properties

A homeomorphism $f\colon X \to X$ is called *topologically transitive* if it has a dense orbit, i.e. if $\{f^n(x) : x \in X\}$ is dense for some $x \in X$. The map f is called *topologically mixing* if, for any nonempty open sets $U, V \subset X$, $f^n(U) \cap V$ is empty for only finitely many n.

When X is a compact metric space, topological transitivity of $f\colon X \to X$ is implied by topological mixing. Indeed, for each n we can cover X with finitely many balls of radius $1/n$ centered at $x_1, \ldots, x_{k_n}$, so that $\mathcal{B} = \{B_{1/n}(x_j) : n \in \mathbb{N}, 1 \leq j \leq k_n\}$ is a countable basis for the topology on X.

For each $U \in \mathcal{B}$, $\bigcup_n f^n(U)$ is open (since f is a homeomorphism) and dense, since for any nonempty $V \subset X$, $f^n(U) \cap V$ is empty for only finitely many n. Therefore $\bigcap_{U\in\mathcal{B}} \bigcup_n f^n(U)$ is dense by the Baire category theorem. In particular, for any $x \in \bigcap_{U\in\mathcal{B}} \bigcup_n f^n(U)$ and $U \in \mathcal{B}$ containing x, we have that the orbit of x is contained in $\bigcup_n f^n(U)$, and is therefore dense. Thus f is topologically transitive.

Proposition 1. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$ be a hyperbolic matrix. Then $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ is topologically mixing.*

Proof. We follow the proof in [KH95, Chapter 1.8]. Let v_1 and v_2 be the two eigendirections of A, with corresponding eigenvalues $\lambda_1 > 1$ and $\lambda_2 = \lambda_1^{-1}$, respectively. Note that λ_1 and λ_2 are irrational (the characteristic polynomial of A is monic with integer coefficients and no integer roots (because A is hyperbolic), and therefore has no rational roots).

Let $U, V \subset \mathbb{T}^2$ be two open balls centered at points p and q, respectively. Inside U we can find a point p' so that the line segment ℓ_1 from p to p' is parallel to v_1, and is therefore stretched by iterating f_A. Since λ_1 is irrational, the line parallel to v_1 has dense image in $\mathbb{T}^2$. Therefore, we can find $N_1 \in \mathbb{N}$ so that $f_A^{N_1}(\ell_1)$ intersects the line through q in the direction of v_2. Hence let $r \in U$ be the point on ℓ_1 so that $r' = f_A^{N_1}(r)$ lies on this line segment.

The idea is that, since $r' = q + tv_2$ for some $t \in \mathbb{R}$, successive iterates of r' under f_A converge toward q, hence only finitely many lie outside of V. Thus $f^n(U) \cap V$ is empty for only finitely many n, since it contains $f^n(r) = f^{n-N_1}(r')$. $\square$

4.2 Anosov diffeomorphisms

4.2.1 Definitions

We now generalize the notion of a hyperbolic toral automorphism to nonlinear functions. Suppose M is a compact Riemannian manifold. An *Anosov diffeomorphism* of M is a diffeomorphism $f : M \to M$ for which we can write T_pM as an internal direct sum $E_p^s \oplus E_p^u$ for each $p \in M$ in such a way that there exist constants $C > 0$ and $0 < \lambda < 1$ for which the following conditions hold at each point $p \in M$:

(1) $df_p(E_p^s) = E_{f(p)}^s$ and $df_p(E_p^u) = E_{f(p)}^u$.

(2) $\|d(f^n)_p(v)\| \leq C\lambda^n \|v\|$ for each $n \in \mathbb{Z}_+$ and $v \in E_p^s$.

(3) $\|d(f^{-n})_p(v)\| \leq C\lambda^{-n} \|v\|$ for each $n \in \mathbb{Z}_+$ and $v \in E_p^u$.

Hyperbolic toral automorphisms $f_A \colon \mathbb{T}^n \to \mathbb{T}^n$ are examples of Anosov diffeomorphisms on $\mathbb{T}^n$. If A is diagonalizable, then we can just take the unstable subbundle E_p^u to be the sum of the eigenspaces for eigenvalues with modulus greater than 1, and the stable subbundle E_p^s to be the sum of the eigenspaces for eigenvalues with modulus less than 1. Even if A is not diagonalizable, it is possible to pick a norm for which A is contracting on generalized eigenspaces E_λ for $|\lambda| < 1$ and expanding on E_λ for $|\lambda| > 1$. See [KH95, Chapter 1.2].

4.2.2 Properties of Anosov diffeomorphisms

Anosov diffeomorphisms have an important property, called structural stability, which roughly says that, if f is Anosov, then any small perturbation of f will remain conjugate to f. To describe this in more detail, we will need to define some terms.

For this section, let M be a compact smooth manifold. Whenever $0 \leq r \leq k \leq \infty$, the set $\mathrm{Diff}^k(M) = \mathrm{Diff}^k(M, M)$ comes equipped with the C^r *topology*, which we define sequentially as follows. First, fix a "nice" finite collection $\mathcal{A}$ of smooth charts whose domains cover M. Given a sequence $(f_n)_{n=1}^\infty$ in $\mathrm{Diff}^k(M)$ and $f \in \mathrm{Diff}^k(M)$, we write $f_n \to f$ if for any pair of smooth charts in $\mathcal{A}$, the coordinate representation of f_n in these charts and its partial derivatives up to order r converge uniformly to the coordinate representation of f and its partial derivatives up to order r, respectively, on a suitable common domain of definition.

Suppose $f, g \in \mathrm{Diff}^k(M)$. We say that f is *topologically conjugate* to g if there exists a homeomorphism $h : M \to M$ such that $f = hgh^{-1}$. Note that topological conjugacy defines an equivalence relation on $\mathrm{Diff}^k(M)$. We say that f is *C^r-structurally stable* if it has a C^r-neighborhood $U \subseteq \mathrm{Diff}^k(M)$ such that every $g \in U$ is topologically conjugate to f.

We can now state the following result:

Theorem 7. *Anosov diffeomorphisms on M are C^1-structurally stable.*

We will only give a sketch of the argument. For the full proof, see [Coo] or [RV]. The first step is to establish the following lemma, which is often referred to as "persistence of hyperbolicity."

Lemma 1. *The set of Anosov diffeomorphisms on M is open in the C^1 topology on* $\mathrm{Diff}^1(M)$*.*

Again, we will only sketch the main idea of proof. Let $f \in \mathrm{Diff}^1(M)$ be Anosov, so that at each point $p \in M$ we have a df-invariant splitting $T_pM = E^s_p \oplus E^u_p$ such that

$$\begin{aligned} \|df^n_p(v)\| &\leq \lambda^n \|v\| \qquad \text{for } v \in E^s_p, \\ \|df^{-n}_p(v)\| &\leq \lambda^n \|v\| \qquad \text{for } v \in E^u_p. \end{aligned}$$

Note here that we assume that $C = 1$ (where C is as in the definition of an Anosov diffeomorphism), but this requires an argument (see Theorem 3.1 in [Coo]). We need to find a neighborhood $\mathcal{N}$ of f such that for any $g \in \mathcal{N}$, we likewise have a splitting $T_pM = \tilde{E}^s_p \oplus \tilde{E}^u_p$ into dg-invariant expanding and contracting subbundles.

Moving forward, we will often suppress the subscripts p from the notation. With respect to the direct sum decomposition $TM = E^s \oplus E^u$ for f, we can write df as a diagonal matrix

$$df = \begin{pmatrix} df_{ss} & \\ & df_{uu} \end{pmatrix},$$

where $df_{ss} = \pi_s \circ df|_{E^s}$ with $\pi_s \colon TM \to E^s$ being the projection map, and df_{uu} is defined similarly. We can also write dg with respect to this same decomposition $TM = E^s \oplus E^u$ for f, to obtain a matrix

$$dg = \begin{pmatrix} dg_{ss} & dg_{su} \\ dg_{us} & df_{uu} \end{pmatrix},$$

where e.g. $dg_{su} = \pi_s \circ dg|_{E^u}$ and the other entries are defined similarly. One can show that the norms of the matrix entries of dg vary continuously with g, so that for g close to f we have that $\|dg_{us}\|$, $\|dg_{su}\|$ are small and $\|dg^{-1}_{uu}\|, \|dg_{ss}\| < \lambda$. Using this, it is possible to construct the subbundles $\tilde{E}^s$ and $\tilde{E}^u$ for g and show that they are dg-invariant and that $TM = \tilde{E}^s \oplus \tilde{E}^u$, hence showing that g is Anosov.

Now we return to the sketch of the proof of Theorem (7). We will make use of two properties of Anosov diffeomorphisms, expansiveness and the shadowing property. A diffeomorphism $f \in \mathrm{Diff}^1(M)$ is said to be *expansive*, with *expansive constant* $r > 0$, if, whenever $x, y \in M$ satisfy $d(f^n(x), f^n(y)) \leq r$ for all $n \in \mathbb{Z}$, then $x = y$. Once one has shown the 'local stable manifold theorem' (see Theorem 2.6 in [Coo]), it follows quickly that any Anosov diffeomorphism is expansive.

A (finite or infinite) sequence of points $\{x_n\}$ is called a δ-*pseudo-orbit* if $d(f(x_n), x_{n+1}) < \delta$ for all pairs x_n, x_{n+1} in the sequence. The sequence is said to be ε-*shadowed* by a point $x' \in M$ if $d(f(x'), x_n) \leq \varepsilon$ for all for all pairs x_n, x_{n+1} in the sequence.

A diffeomorphism $f \in \mathrm{Diff}^1(M)$ is said to satisfy the *shadowing property* if there is some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, we can pick $\delta > 0$ small enough so that every δ-pseudo-orbit for f is ε-shadowed by a genuine orbit of some $x' \in M$. It is possible to show that any Anosov diffeomorphism has the shadowing property, and moreover we can use expansiveness to show that the point x' is unique.

Using persistence of hyperbolicity, we in fact have that these two properties persist in small neighborhoods of an Anosov diffeomorphism $f \in \mathrm{Diff}^1(M)$.

Now let us fix some $f \in \mathrm{Diff}^1(M)$ and pick $\mathcal{N}_0$ to be a small neighborhood of f in which expansivity and the shadowing property persist. Pick some $g \in \mathcal{N}$. We construct a homeomorphism $h\colon M \to M$ conjugating f and g using the shadowing property. Given $x \in M$, we know that the orbit of x under g is a pseudo-orbit for f because f and g are close in the C^0 metric on M. Therefore, the shadowing property tells us that there exists a unique point $x' \in M$ which shadows the orbit of x under g. Define $h\colon M \to M$ by setting $h(x) = x'$.

Note that if x' shadows the orbit of x under g, then $f(x')$ shadows the orbit of $g(x)$ under g. Hence we have that $h(g(x)) = f(x')$, which shows that

$$\begin{aligned} h^{-1} \circ f \circ h(x) &= h^{-1}(f(x')) \\ &= g(x), \end{aligned}$$

so h conjugates f and g.

We can again use the shadowing property of g to show that h is continuous. Let $\{x_n\}$ be a sequence in M converging to $x \in M$. Since M is compact, after passing to a subsequence we know that $\{h(x_n)\}$ converges to some $y \in M$. Notice that

$$d(f^i(h(x_n)), g^i(x)) \leq d(f^i(h(x_n)), g^i(x_n)) + d(g^i(x_n), g^i(x)).$$

Taking $i \to \infty$, the first term is bounded by a fixed $\varepsilon > 0$ by the shadowing property, and the second quantity tends to zero by continuity of g. Now taking $n \to \infty$, we see that the orbit of x under g is shadowed by $\lim_{n\to\infty} h(x_n) = y$, so $h(x) = y$, proving continuity. One argues in a similar spirit using the shadowing property that h is a homeomorphism, and this completes the proof.

Another property of Anosov diffeomorphisms is that they are closed under conjugation: To see this, let γ and γ' be smooth diffeomorphisms on a compact manifold M, and suppose that γ is Anosov. If $\gamma' \in \Gamma$ then $d(\gamma'\gamma\gamma'^{-1}) = d(\gamma')d(\gamma)d(\gamma')^{-1}$. Let $A = d(\gamma)$ and $Q = d(\gamma')$ (note that $d(\gamma) = d_p(\gamma)$ and $d(\gamma') = d_q(\gamma')$ depend on $p, q \in M$, but we drop the subscript for notational convenience). If $\|Av\| < \lambda\|v\|$ then

$$\begin{aligned} \|(QAQ^{-1})^n(Qv)\| = \|QA^nQ^{-1}(Qv)\| &= \|QA^nv\| \\ &\le \|Q\|\|A^nv\| \\ &< \|Q\|\lambda^n\|v\|, \end{aligned}$$

where

$$\|Q\| = \|d_q(\gamma')\| = \sup_{\|w\|=1} \|d_q(\gamma')w\|$$

is bounded above as q varies over points in M since M is compact. Similarly if $\|A^{-1}v\| < \lambda\|v\|$ then we have that $\|(QAQ^{-1})^{-n}(Qv)\| < \|Q\|\lambda^n\|v\|$.

Let E^u and E^s be the stable and unstable subbundles for γ, and define $\tilde{E}^u := d(\gamma')(E^u) = Q(E^u)$ and $\tilde{E}^s := d(\gamma')(E^s) = Q(E^s)$. Then the above observation shows that

$$\begin{aligned} \|d(\gamma'\gamma\gamma'^{-1})^n(w)\| &= \|d(\gamma'\gamma\gamma'^{-1})^n(QQ^{-1}w)\| \\ &\le \|Q\|\lambda^n\|Q^{-1}w\| \\ &\le \|Q\|\|Q^{-1}\|\lambda^n\|w\| \\ &= C\lambda^n\|w\| \end{aligned}$$

for all $w \in \tilde{E}^s$, where $C = \sup_p\{\|Q\|\|Q^{-1}\|\}$ is a fixed constant. We also know that $\tilde{E}^s$ is $d(\gamma'\gamma\gamma'^{-1})$-invariant, since

$$d(\gamma'\gamma\gamma'^{-1})(\tilde{E}^s) = d(\gamma'\gamma\gamma'^{-1}) \circ d(\gamma')(E^s) = d(\gamma') \circ d(\gamma)(E^s) = d(\gamma')(E^s) = \tilde{E}^s$$

by the $d(\gamma)$-invariance of E^s. Similarly $\tilde{E}^u$ is $d(\gamma'\gamma\gamma'^{-1})$-invariant and for all $w \in \tilde{E}^u$, $\|d(\gamma'\gamma\gamma'^{-1})^{-n}(w)\| \le C\lambda^n\|w\|$.

Moreover, by similar reasoning we see that the inverse map $w \mapsto Q^{-1}w$ also sends $\tilde{E}^s$ into E^s and $\tilde{E}^u$ into E^u, so we in fact get isomorphisms $E^s \cong \tilde{E}^s$ and $E^u \cong \tilde{E}^u$. Since $TM = E^u \oplus E^s$, this therefore implies that $TM \cong \tilde{E}^u \oplus \tilde{E}^s$ since $d(\gamma')$ is an isomorphism. Hence $\gamma'\gamma\gamma'^{-1}$ is Anosov.

4.3 Anosov actions

4.3.1 Definitions

A *smooth group action* of a group Γ on a manifold M is a homomorphism $\alpha : \Gamma \to \mathrm{Diff}^\infty(M)$. For example, $\mathrm{SL}(n, \mathbb{Z})$ acts on $\mathbb{T}^n$ by the map $A \mapsto f_A$. Note that the notion of a smooth group action generalizes the notion of a single smooth diffeomorphism f acting on a manifold M, since one can think of this as a $\mathbb{Z}$ action (via the map $n \mapsto f^n$).

One says that an action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ is *Anosov* if $\alpha(\gamma)$ is Anosov for some $\gamma \in \Gamma$. If Γ is finitely generated, then there is a notion of structural stability for actions $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$. If $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle$ is a finitely generated group then we can define the C^r distance between two actions $\alpha, \alpha'\colon \Gamma \to \mathrm{Diff}^\infty(M)$ to be the maximum of the C^r distances between $\alpha(\gamma_i)$ and $\alpha'(\gamma_i)$ for each generator γ_i.

We say that $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ is *topologically rigid* if there exists some $\varepsilon > 0$ such that, for any $\alpha'\colon \Gamma \to \mathrm{Diff}^\infty(M)$ within an ε-ball of α in the C^1 topology, we can find a homeomorphism $H\colon M \to M$ such that $H \circ \alpha'(\gamma) \circ H^{-1} = \alpha(\gamma)$ for all $\gamma \in \Gamma$. Notice that H cannot depend on γ, so this is stronger than requiring each $\alpha'(\gamma)$ be conjugate to $\alpha(\gamma)$ by any homeomorphism H_γ.

We say that $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ is *C^r-deformation rigid* if, whenever we have a continuous path α_t ($t \in [0,1]$) of C^r actions on M with $\alpha_0 = \alpha$ and which are contained in an ε-ball around M (for ε small enough), we can find a family of C^r diffeomorphisms $H_t\colon M \to M$ ($t \in [0,1]$) such that $H_0 = \mathrm{id}_M$ and $H_t \circ \alpha'_t(\gamma) \circ H_t^{-1} = \alpha_t(\gamma)$ for all $\gamma \in \Gamma$ and all $t \in [0,1]$. If $r = 0$ then we say that α is *topologically deformation rigid.*

4.3.2 Rigidity

As we have seen, individual Anosov diffeomorphisms are structurally stable. On the other hand, as the following example of Hurder demonstrates (Theorem 7.22 in [Hur92]), one can have Anosov group actions which are *not* topologically deformation rigid.

Theorem 8 (Hurder, 1990)**.** *There exists an analytic family $\{\varphi_t : 0 \le t \le 1\}$ of volume-preserving, real analytic actions of* $\mathrm{SL}(2,\mathbb{Z})$ *on* $\mathbb{T}^2$*, with $\varphi_0 = \varphi$ the standard action, such that φ_t is not topologically conjugate to φ for all* $0 \le t \le 1$.

Here, the standard action φ of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$ is the action by matrix multiplication, i.e. $\varphi(A)$ is the map $x + \mathbb{Z}^2 \mapsto Ax + \mathbb{Z}^2$ (we will often just write A to mean $\varphi(A)$). For clarity and to motivate our later work, we recount the proof here.

The key idea is to use the fact that $\mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

which have order 4 and 6, respectively. Furthermore, these matrices have very few relations between them: the only nontrivial relation is that $A^2 = B^3 = -I$. Therefore, $\mathrm{SL}(2,\mathbb{Z})$ looks like the free product of the groups $\mathbb{Z}/(4) \cong \langle A \rangle$ and $\mathbb{Z}/(6) \cong \langle B \rangle$, with the additional relation $A^2 = B^3 = -I$ (in other words, $\mathrm{SL}(2,\mathbb{Z})$ can be written as an amalgamated product $\mathrm{SL}(2,\mathbb{Z}) \cong \mathbb{Z}/(4) *_{\mathbb{Z}/(2)} \mathbb{Z}/(6)$). This means we have some freedom in how to perturb the generators A and B, since we only have to preserve the relation $A^2 = B^3 = -I$.

Hurder constructs a one-parameter family $\{\varphi_t : 0 \le t \le 1\}$ of deformations of the standard action $\varphi_0 = \varphi$ of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$ as follows. Consider the vector field $\overrightarrow{Z}_1 = x\,\partial_y - y\,\partial_x$, which is a counterclockwise rotation around the origin. We restrict $\overrightarrow{Z}_1$ to a small ball around the origin by taking a smooth bump function $\psi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\psi) \subset B_{10^{-4}}(0)$ and $\psi|_{B_{10^{-5}}(0)} \equiv 1$, and form the vector field

$$\overrightarrow{Z}_\psi = \psi(x^2+y^2)\overrightarrow{Z}_1.$$

We now form the vector field Z to be

$$Z = \sum_{(n,m)\in\mathbb{Z}^2} DT_{(n+1/2,m)}\overrightarrow{Z}_\psi,$$

where $DT_{(n+1/2,m)}\overrightarrow{Z}_\psi$ is the translate of $\overrightarrow{Z}_\psi$ to be centered at $(n+1/2,m)$. Note that this sum is well defined since the supports of the translates do not overlap.

Let $F(t)$ be the rotational flow of this vector field on $\mathbb{R}^2$, which descends to a flow on $\tilde{F}(t)$. We deform the standard action φ of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$ by perturbing the action of A:

$$\begin{aligned}
\varphi_t(A) &= \tilde{F}^{-1}(t)\circ\varphi(A)\circ\tilde{F}(t),\\
\varphi_t(B) &= B.
\end{aligned}$$

In order to check that this really defines an action of $\mathrm{SL}(2,\mathbb{Z})$, we need to verify that $\varphi_t(A)^2 = \mathrm{id}_{\mathbb{R}^2}$ and $\varphi_t(B)^3 = \mathrm{id}_{\mathbb{R}^2}$. The latter identity is clear, and for the former we have that

$$\begin{aligned}
\varphi_t(A)^2 &= (\tilde{F}^{-1}(t)\circ\varphi(A)\circ\tilde{F}(t))(\tilde{F}^{-1}(t)\circ\varphi(A)\circ\tilde{F}(t))\\
&= \tilde{F}^{-1}(t)\circ\varphi(A^2)\circ\tilde{F}(t)\\
&= \mathrm{id}_{\mathbb{R}^2},
\end{aligned}$$

where the last line follows because $A^2 = -I$, which is a rotation through π, and thus $\varphi(A^2)\circ\tilde{F}(t) = \tilde{F}(t)$ since this rotation preserves $\tilde{F}$.

Thus, we have a family $\{\varphi_t : 0 \le t \le 1\}$ of deformations of φ. The next step is to show that φ_t cannot be conjugated to φ if $t \neq 0$, which is the content of the next proposition.

Proposition 2. *If there exists a homeomorphism $H\colon \mathbb{T}^2 \to \mathbb{T}^2$ conjugating φ_t to φ_0, then $t = 0$.*

Proof. H must fix the origin, so it admits a unique lift $\hat{H}\colon \mathbb{R}^2 \to \mathbb{R}^2$ which fixes the origin and conjugates $\hat{\varphi}_t$ to $\hat{\varphi}_0$. Thus,

$$\hat{H}\circ A\circ\hat{H}^{-1} = \hat{\varphi}_t(A), \tag{1}$$

$$\hat{H}\circ B\circ\hat{H}^{-1} = B. \tag{2}$$

Since $\varphi_t(A) = \tilde{F}(t)^{-1}\circ\varphi(A)\circ\tilde{F}(t)$, the first equality can be written as

$$A = G^{-1}\circ A\circ G \tag{3}$$

with $G = F(t) \circ \hat{H}$.

We claim that

$$C^{-1} \circ F(t) \circ \hat{H} \circ C = \hat{H} \tag{4}$$

is true when restricted to the x-axis. To see this, note that because A commutes with $F(t) \circ \hat{H}$ and B commutes with $\hat{H}$, we have that

$$\begin{aligned} C^{-1} \circ F(t) \circ \hat{H} \circ C &= B^{-2} \circ A \circ F(t) \circ \hat{H} \circ A^{-1} \circ B^2 \\ &= B^{-2} \circ F(t) \circ \hat{H} \circ B^2 \\ &= B^{-2} \circ F(t) \circ B^2 \circ \hat{H}. \end{aligned}$$

We claim that image of the x-axis under $\hat{H}$ is reasonably close to the x-axis. In particular, this means that $B^2 \circ \hat{H}$ sends the x-axis to a neighborhood of the y-axis, so $F(t)$ acts trivially on it. Hence we have that $B^{-2} \circ F(t) \circ B^2 \circ \hat{H}(x,0) = B^{-2} \circ B^2 \circ \hat{H}(x,0) = \hat{H}(x,0)$ as desired.

To justify this claim, first note that, by inverting both sides of equation (1) and using that $\varphi_t^{-1}(A) = \varphi_t(A^{-1})$, it follows that $\hat{H}A^{-1} = \varphi_t(A^{-1})\hat{H}$. Therefore

$$\hat{H}A^{-1}B^2 = \varphi_t(A^{-1})\hat{H}B^2 = \varphi_t(A^{-1})B^2\hat{H}.$$

Since $A^{-1}B^2$ preserves the x-axis, the above equation shows that $\hat{H}(x,0) = \varphi_t(A^{-1})B^2\hat{H}(x,0)$, i.e. that $\hat{H}(x,0) \in (\varphi_t(A^{-1})B^2 - \mathrm{id}_{\mathbb{R}^2})^{-1}((0,0))$. Because $A^{-1}B^2(x,y) = (x-y,y)$, the set $(A^{-1}B^2 - \mathrm{id}_{\mathbb{R}^2})^{-1}((0,0))$ is precisely the x-axis. Note that φ_t is a continuous deformation of φ, so we can find t small enough so that $\|\varphi_t - \varphi_0\|_{\sup} < \varepsilon$ for some small ε. Then $(\varphi_t(A^{-1})B^2 - \mathrm{id}_{\mathbb{R}^2})^{-1}((0,0))$ is contained in $(A^{-1}B^2 - \mathrm{id}_{\mathbb{R}^2})^{-1}(B_\varepsilon(0))$, which is equal to the strip $S_\varepsilon = \{(x,y) \in \mathbb{R}^2 : |y| < \varepsilon\}$. Thus $\hat{H}(x,0) \in S_\varepsilon$, showing that image of the x-axis under $\hat{H}$ is reasonably close to the x-axis.

Then, restricting (4) to the x-axis gives us that

$$F(t) \circ \hat{H}_1(x,0) = C \circ \hat{H}_1(x,0). \tag{5}$$

We know that $\hat{H}$ fixes the origin. On the other hand, if $t \neq 0$ then $\hat{H}$ cannot map the entire x-axis into itself, for then (4) cannot hold for $(x,0)$ with $\hat{H}(x,0)$ lying in the support of $F(t)$, since then $F(t)$ will perturb these points vertically while C preserves horizontal lines. Thus, by continuity we can find $(x',0)$ such that $\hat{H}(x',0) = (x'',c)$ for $c > 0$ small. But then (5) cannot hold, for C is a leftward shearing motion and, when c and t are sufficiently small, $F(t)$ is not. $\square$

5 Nilmanifolds

There is a natural question: what kind of spaces explicit Anosov actions? We had seen earlier that the torus has Anosov $\mathbb{Z}$-actions using hyperbolic toral automorphisms (e.g. cat map), and one of our results is that the surface groups have an Anosov action on $\mathbb{T}^3$. It turns out that nilmanifolds are the spaces that we want to consider. Intuitively, they are generalization of $\mathbb{T}^d$. Our goal in this section is to explore whether we can generalize our result of an Anosov surface group action on $\mathbb{T}^3$ to an action on nilmanifolds. Before delving into the notion of nilmanifolds and how we can construct a surface group action on them, we begin with a review of Lie theory.

5.1 Lie Theory

5.1.1 Review of Lie Groups and Lie Algebras

A *Lie group* G is a smooth manifold with group structure in which multiplication $\times : G \times G \to G$ and inverse $(-)^{-1} : G \to G$ are smooth maps.

A trivial example is $\mathbb{R}^d$ with addition. A more standard example is the general linear group $\mathrm{GL}(n, \mathbb{R})$. It is a smooth n^2-dimensional manifold, and we can give it group structure by matrix multiplication and matrix inversion, which can be shown to be smooth. We will be interested in closed subgroups of $\mathrm{GL}(n, \mathbb{R})$. They are Lie groups from Theorem 20.12 on [Lee03], and we call them *matrix Lie groups*. A example of this is the *Heisenberg group* $H_3(\mathbb{R}) \leq GL(3, \mathbb{R})$, which consists of matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

for $a, b, c \in \mathbb{R}$. This is a closed subgroup of $\mathrm{GL}(3, \mathbb{R})$, making it a 3-dimensional Lie group. The Heisenberg group will be a recurring example throughout this section.

A *Lie algebra* $\mathfrak{g}$ is a vector space with a operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying

1. Bilinearity: $[\lambda_1 u + \lambda_2 v, w] = \lambda_1[u, w] + \lambda_2[v, w]$ and $[w, \lambda_1 u + \lambda_2 v] = \lambda_1[w, u] + \lambda_2[w, v]$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u, v, w \in \mathfrak{g}$,
2. Alternativity: $[u, u] = 0$ for $u \in \mathfrak{g}$,
3. Jacobi Identity: $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ for $u, v, w \in \mathfrak{g}$,
4. Anti-commutativity: $[u, v] = -[v, u]$ for $u, v \in \mathfrak{g}$.

We shall assume that all Lie algebras are finite-dimensional and are over $\mathbb{R}$. These two objects are very related, particularly by the Lie-Group-Lie-Algebra correspondence. This gives us a one-to-one way to correspond simply connected Lie groups and Lie algebras with nice properties. Namely, given a Lie group, we can consider its tangent space at identity and assign it a certain bracket to make a Lie algebra. When the Lie group is simply-connected, this association is unique. We will not go into too much detail and refer the reader to plenty of references along the way, see [Hal03] and [Lee03] for more formality.

Of particular interest is the correspondence for matrix Lie groups. For a matrix Lie group G, we can define its associated Lie algebra $\mathfrak{g}$ as follows. Consider the map $\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined on n-by-n real matrices given by

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}. \tag{7}$$

We can see that series converges for all $X \in M_n(\mathbb{R})$, so map is well-defined. Then the Lie algebra $\mathfrak{g}$ consists of n-by-n real matrices M in which $\exp(tM) \in G$ for all $t \in \mathbb{R}$, see Theorem 3.20 on [Hal03]. That is, $\mathfrak{g} = \{X \in M_n(\mathbb{R}) : \exp(tX) \in G, \forall t \in \mathbb{R}\}$. We call the restricted map $\exp : \mathfrak{g} \to G$ the *exponential map* from the Lie algebra $\mathfrak{g}$ to the Lie group G.

As an example, the Lie algebra $\mathfrak{h}_3(\mathbb{R})$ of the Lie group $H_3(\mathbb{R})$ consists of matrices

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

for $a, b, c \in \mathbb{R}$, with Lie brackets $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{h}_3(\mathbb{R})$, see Proposition 3.26 on [Hal03] . Note that for $X \in \mathfrak{h}_3(\mathbb{R})$, we have $X^n = 0$ for $n \geq 3$, so its exponential is just $\exp(X) = I + X + X^2/2$.

5.1.2 Nilpotent Groups and Lie Algebras

Before proceeding, we define the notion of nilpotent groups and Lie algebras. A group G is *nilpotent* if we have a series of normal subgroups

$$\{1\} = Z_0 \lhd Z_1 \lhd \ldots \lhd Z_n = G$$

where $Z_1 = Z(G)$ and $Z_{i+1}/Z_i = Z(G/Z_i)$, and we say G is *n-step nilpotent*, where n is the smallest such. Note that G is 1-step nilpotent if and only if it is abelian. Intuitively, these groups are almost abelian, and the smaller n is, the more abelian it is.

A Lie algebra $\mathfrak{g}$ is nilpotent if we have a series of subalgebras

$$\mathfrak{g} = \mathfrak{g}_0 \geq \ldots \geq \mathfrak{g}_{n-1} \geq \mathfrak{g}_n = \{0\}$$

where $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$, and we say $\mathfrak{g}$ is *n-step nilpotent*, where n is the smallest such. We remark that there are several equivalent ways to define nilpotency for groups; in particular, the lower central series definition parallels the notion for Lie algebra.

For instance, the Heisenberg group $H_3(\mathbb{R})$ and its Lie algebra are 2-step nilpotent (as a group and a Lie algebra, respectively). Indeed, its center $Z(H_3(\mathbb{R}))$ consists of matrices in the form (6) with $a = b = 0$ and $c \in \mathbb{R}$. One can then verify computationally that $Z(H_3(\mathbb{R}))$ is abelian, however $H_3(\mathbb{R})$ itself is not abelian. Similarly, we can see that $[\mathfrak{h}_3(\mathbb{R}), \mathfrak{h}_3(\mathbb{R})]$ consists of matrices in the form (8) with $a = b = 0$ and $c \in \mathbb{R}$, and, in turn, it can be shown that $[\mathfrak{h}_3(\mathbb{R}), [\mathfrak{h}_3(\mathbb{R}), \mathfrak{h}_3(\mathbb{R})]] = \{I\}$.

5.2 Nilmanifolds

Let G be a simply-connected Lie group. Consider a closed subgroup $H \leq G$. The left coset space G/H, which consists of $gH = \{gh : h \in H\}$ for $g \in G$, is a smooth manifold with the quotient topology, see Proposition 21.17 on [Lee03]. With this, we define a *lattice* Γ of G to be a subgroup of G in which Γ is discrete (every point is isolated) and G/Γ is compact. We remark that Γ is not generally a normal subgroup, so G/Γ may not be a Lie group.

Given a simply-connected nilpotent Lie group G and a lattice $\Gamma \leq G$, we call the quotient G/Γ a *nilmanifold*.

A basic example is to take $G = \mathbb{R}^d$, which is simply-connected and abelian (so nilpotent), with $\Gamma = \mathbb{Z}^d$, which is certainly discrete. Then $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is compact, so it is a nilmanifold. A more non-trivial example of a nilmanifold is the following: let $G = \mathfrak{h}_3(\mathbb{R})$ be the Heisenberg group. As stated previously, this is a (nonabelian) nilpotent Lie group. Identifying it with $\mathbb{R}^3$ tells us that it is simply connected. Now, consider the subset $\Gamma_{\mathfrak{h}_3(\mathbb{R})}$, consisting of matrices of the form (6) where we take coefficients a, b, c from $\mathbb{Z}$. One can show that $\Gamma_{\mathfrak{h}_3(\mathbb{R})}$ is a lattice of $\mathfrak{h}_3(\mathbb{R})$, making $\mathfrak{h}_3(\mathbb{R})/\Gamma_{\mathfrak{h}_3(\mathbb{R})}$ a nilmanifold.

It turns out that the nilmanifold $\mathfrak{h}_3(\mathbb{R})/\Gamma_{\mathfrak{h}_3(\mathbb{R})}$ defined above does not exhibit any Anosov diffeomorphisms, see Exercise 17.3.4 on [KH95]. We shall now see an example of one that does.

5.3 Anosov Diffeomorphism on Nilmanifolds

Here, we will describe the construction of a particular nilmanifold with an Anosov diffeomorphism acting on it. This example is due to Smale and Borel, see [Sma67] on page 762. However, the source contains a mistake regarding the lattice not being a subgroup, so we will follow a corrected construction by Burns and Wilkinson on page 95 of [BW08].

Let us denote $H = \mathfrak{h}_3(\mathbb{R})$ and $\mathfrak{h} = \mathfrak{h}_3(\mathbb{R})$ for this subsection. The Lie group we shall be dealing with is $G = H \times H$. This is simply-connected and nilpotent as direct sum preserves those properties. We can view G as 6-by-6 matrices of the form

$$\begin{pmatrix} 1 & a_1 & c_1 & 0 & 0 & 0 \\ 0 & 1 & b_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a_2 & c_2 \\ 0 & 0 & 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

for $a_i, b_i, c_i \in \mathbb{R}$. The Lie algebra of G is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, and similarly its exponential map $\exp_{\mathfrak{g}} = \exp_{\mathfrak{h}} \oplus \exp_{\mathfrak{h}}$, see Exercise 3.9.5 in [Hal03]. In turn, we can represent it as 6-by-6 matrices

$$\begin{pmatrix} 0 & a_1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & c_2 \\ 0 & 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

for $a_i, b_i, c_i \in \mathbb{R}$. It is generated by X_1, Y_1, Z_1, X_2, Y_2, and Z_2, which are the matrices of the form (10) with zero entries everywhere except a one at a_1, b_1, c_1, a_2, b_2, and c_2, respectively.

Let σ be the field automorphism of $\mathbb{Q}(\sqrt{3})$ given by $a + b\sqrt{3} \mapsto a - b\sqrt{3}$ for $a, b \in \mathbb{Q}$, and define

$$\Gamma_0 = \left\{ \begin{pmatrix} 0 & u & w/2 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(u) & \sigma(w/2) \\ 0 & 0 & 0 & 0 & 0 & \sigma(v) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : u, v, w \in \mathbb{Z}[\sqrt{3}] \right\}.$$

Next, define

$$\begin{aligned} \Gamma &= \exp(\Gamma_0) \\ &= \left\{ \begin{pmatrix} 1 & u & (w+uv)/2 & 0 & 0 & 0 \\ 0 & 1 & v & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sigma(u) & \sigma((w+uv)/2) \\ 0 & 0 & 0 & 0 & 1 & \sigma(v) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : u, v, w \in \mathbb{Z}[\sqrt{3}] \right\} \\ &\subseteq G. \end{aligned}$$

For convenience, let us relabel the elements of G via the identification

$$\begin{pmatrix} 1 & u & w & 0 & 0 & 0 \\ 0 & 1 & v & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & z \\ 0 & 0 & 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix}$$

for each $u, v, w, x, y, z \in \mathbb{R}$. The matrix product of two elements of Γ is then given by the formula

$$\begin{pmatrix} u \\ v \\ (w+uv)/2 \\ \sigma(u) \\ \sigma(v) \\ \sigma((w+uv)/2) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ (z+xy)/2 \\ \sigma(x) \\ \sigma(y) \\ \sigma((z+xy)/2) \end{pmatrix} = \begin{pmatrix} u+x \\ v+y \\ ((u+x)(v+y)+w+z+uy-xv)/2 \\ \sigma(u+x) \\ \sigma(v+y) \\ \sigma((u+x)(v+y)+w+z+uy-xv) \end{pmatrix}$$

for $u, v, w, x, y, z \in \mathbb{Z}[\sqrt{3}]$. From this, it is evident that Γ is closed under matrix multiplication and matrix inversion, and we see from its definition that Γ contains the identity matrix. Therefore, Γ is a subgroup of G. It is also discrete and cocompact in G, and so the quotient G/Γ is a nilmanifold.

We will now define an Anosov diffeomorphism of G/Γ. Let $\lambda = 2+\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$, and note that $\lambda^{-1} = \sigma(\lambda) = 2-\sqrt{3}$. Define an $\mathbb{R}$-linear isomorphism $f : \mathfrak{g} \to \mathfrak{g}$ by

$$\begin{aligned} f(X_1) &= \lambda X_1 \\ f(Y_1) &= \lambda^2 Y_1 \\ f(Z_1) &= \lambda^3 Z_1 \\ f(X_2) &= \sigma(\lambda) X_2 \\ f(Y_2) &= \sigma(\lambda)^2 Y_2 \\ f(Z_2) &= \sigma(\lambda)^3 Z_2. \end{aligned}$$

Note that f preserves the commutator Lie bracket on the basis $\{X_1, Y_1, Z_1, X_2, Y_2, Z_2\}$ for $\mathfrak{g}$, and so it is a Lie algebra automorphism of $\mathfrak{g}$. As before, let us relabel the elements of $\mathfrak{g}$ via the identification

$$\begin{pmatrix} 0 & u & w & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & z \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix}.$$

The values of f at matrices in Γ_0 are given by

$$f \begin{pmatrix} 0 & u & w/2 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(u) & \sigma(w/2) \\ 0 & 0 & 0 & 0 & 0 & \sigma(v) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda u & \lambda^3 w/2 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(\lambda u) & \sigma(\lambda^3 w/2) \\ 0 & 0 & 0 & 0 & 0 & \sigma(\lambda^2 v) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this, we see that f takes Γ_0 onto Γ_0.

Let F be the Lie group automorphism of G that induces f. The commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{F} & G \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{g} & \xrightarrow[f]{} & \mathfrak{g} \end{array}$$

and the bijectivity of the vertical maps (due to G being nilpotent and simply-connected) implies that $F(\Gamma) = \exp\big(f(\exp^{-1}(\Gamma))\big) = \exp(f(\Gamma_0)) = \exp(\Gamma_0) = \Gamma$, i.e., Γ is F-invariant. It follows that F descends to the quotient, yielding a map $\overline{F} : G/\Gamma \to G/\Gamma$ given by $F(A\Gamma) = F(A)\Gamma$ for each $A \in G$. To see that this map is well-defined, suppose $A, B \in G$ satisfy $A\Gamma = B\Gamma$. Then $B^{-1}A\Gamma = \Gamma$, so $B^{-1}A \in \Gamma$. Since Γ is F-invariant, it follows that $F(B)^{-1}F(A) = F(B^{-1}A) \in \Gamma$, i.e., $F(A)\Gamma = F(B)\Gamma$ as was desired. A similar argument shows that $\overline{F}$ is injective, and hence bijective. The smoothness of F implies the smoothness of $\overline{F}$, and the smoothness of F^{-1} implies the smoothness of $\overline{F}^{-1}$. Therefore, $\overline{F}$ is a diffeomorphism of G/Γ.

6 Main Results

6.1 Action of genus n surface group on 3-torus

We find isometries in terms of the generators of $D(3,3,4)$ which induce side pairings on a hyperbolic octagon in the hyperbolic plane. We then use Poincare's Theorem (Theorem 3) to conclude that this gives us a presentation of $\pi_1(\Sigma_2)$ in $D(3,3,4)$. Finally, we combine these results with Theorem 4 to obtain a family of Anosov surface group actions on $\mathbb{T}^3$.

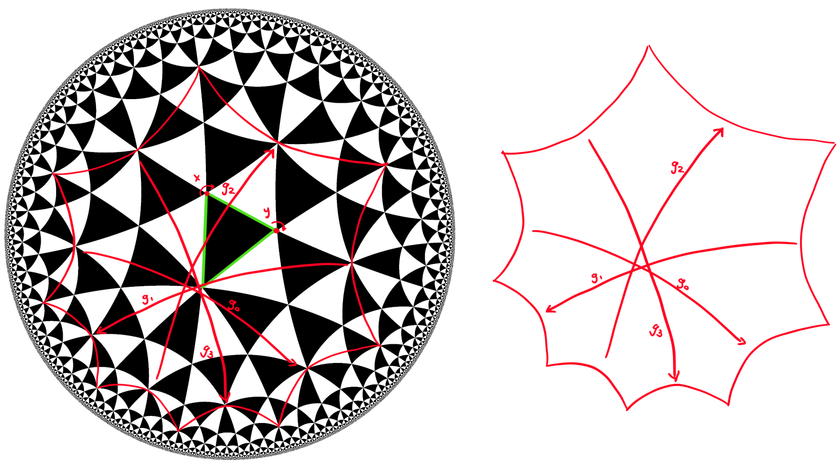


Figure 5: The fundamental polygon and side pairings for the embedding $\pi_1(\Sigma_2) \hookrightarrow D(3,3,4)$

Theorem 9. *There is an embedding of* $\pi_1(\Sigma_2) = \langle g_0, g_1, g_2, g_3 \mid g_0 g_1 g_2 g_3 g_0^{-1} g_1^{-1} g_2^{-1} g_3^{-1} = 1\rangle$ *into* $D(3,3,4) = \langle x, y \mid x^3 = y^3 = (xy)^4 = 1\rangle$ *given by*

$$\begin{aligned} g_0 &= (y^{-1}xy^{-1})(x^{-1}yx^{-1}) \\ g_1 &= (xyx^{-1})(yxy^{-1}) \\ g_2 &= (yxy)x^{-1}(yxy) \\ g_3 &= (y^{-1}x^{-1}y)(x^{-1}y^{-1}x). \end{aligned}$$

Proof. The isometries g_0, g_1, g_2, and g_3 induce side pairings on the hyperbolic octagon, as pictured in figure 5. As one can see from the figure, there is only only elliptic cycle associated to the word $g_0 g_1 g_2 g_3 g_0^{-1} g_1^{-1} g_2^{-1} g_3^{-1}$. This satisfies the elliptic cycle condition with cycle number 1. Thus, Poincare's Theorem tells us that $\langle g_0, g_1, g_2, g_3\rangle < D(3,3,4)$ has the presentation

$$\langle g_0, g_1, g_2, g_3 \mid g_0 g_1 g_2 g_3 g_0^{-1} g_1^{-1} g_2^{-1} g_3^{-1} = 1\rangle \cong \pi_1(\Sigma_2).$$

□

Theorem 10. *There is an embedding of* $\pi_1(\Sigma_2)$ *(using the same presentation as in Theorem 9) into* $D(4,4,4)$ *given by*

$$\begin{aligned} g_0 &= x^2yxyx \\ g_1 &= x^2y^2 \\ g_2 &= x^{-1}yxy \\ g_3 &= x^{-1}y^2x^{-1}. \end{aligned}$$

Remark 6. There is an embedding of $D(4,4,4) = \langle a, b | a^4 = b^4 = (ab)^4 = 1\rangle$ into $D(3,3,4)$ given by $a \mapsto xy$, $b \mapsto yx$. [Math stack exchange] Combined with this remark, Theorem 9 gives an embedding of $\pi_1(\Sigma_2)$ in $D(3,3,4)$.

Proof. The proof technique is very similar to that for Theorem 9, so we omit it for brevity. □

By combining the above results with Theorem 5, we have the following result.

Theorem 11. *There is an embedding of*

$$\pi_1(\Sigma_g) = \langle a^{g-1}, b, c, d, aca^{-1}, ada^{-1}, ..., a^{g-2}ca^{-(g-2)}, a^{g-2}da^{-(g-2)}\rangle$$

into $D(3,3,4) = \langle x, y \mid x^3 = y^3 = (xy)^4 = 1\rangle$ *given by*

$$\begin{aligned} a &= y^{-1}xy^{-1}xyx^{-1}y^{-1} \\ b &= yxy^{-1}xy^{-1}x^{-1}yxy^{-1} \\ c &= yxyx^{-1}y^{-1}x^{-1}y^{-1}x \\ d &= (y^{-1}x^{-1}y^{-1})x(y^{-1}x^{-1}y^{-1}). \end{aligned}$$

Recall Theorem 4 from [LRT11], which gives us a family $\{\rho_t \colon D(3,3,4) \hookrightarrow \mathrm{SL}(3,\mathbb{Z})\}_{t\in\mathbb{Z}}$ of Zariski dense representations of $D(3,3,4)$ into $\mathrm{SL}(3,\mathbb{Z})$. We can compose these representations with the above embeddings $\pi_1(\Sigma_g) \hookrightarrow D(3,3,4)$ to obtain representations

$$\rho_{g,t} \colon \pi_1(\Sigma_g) \hookrightarrow D(3,3,4) \hookrightarrow \mathrm{SL}(3,\mathbb{Z})$$

of $\pi_1(\Sigma_g)$ in $\mathrm{SL}(3,\mathbb{Z})$. Since each representation is Zariski dense and $\pi_1(\Sigma_g)$ is finite-index in $D(3,3,4)$, each of the representations of $\pi_1(\Sigma_g)$ in $\mathrm{SL}(3,\mathbb{Z})$ will have Anosov elements. For each of these representations, we can obtain explicit matrices which generate $\pi_1(\Sigma_g)$ in $\mathrm{SL}(3,\mathbb{Z})$ by substituting the matrices

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 2-t+t^2 & 3+t^2 \\ 0 & -2+2t-t^2 & -1+t-t^2 \\ 0 & 3-3t+t^2 & (-1+t)^2 \end{pmatrix}$$

into the expressions given by Theorem 11. Thus, we obtain our main result:

Theorem 12. *For* $g \geq 2$*, we obtain an explicit family of Anosov actions of* $\pi_1(\Sigma_g)$ *on* $\mathbb{T}^3$*, given by*

$$a \cdot (x + \mathbb{Z}^3) = \rho_{g,t}(a)x + \mathbb{Z}^3$$

for $a \in \pi_1(\Sigma_g)$ *and* $x + \mathbb{Z}^3 \in \mathbb{T}^3$*.*

6.2 Faithful Zariski dense representation of $D(2,3,\infty) \cong PSL(2,\mathbb{Z})$ in $SL(3,\mathbb{Z})$

Slightly adapting the presentation given in [Con22], we may write

$$PSL(2,\mathbb{Z}) = \langle x, y \mid x^2 = y^3 = 1\rangle$$

where

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Theorem 13. *There is a 3-dimensional integral faithful Zariski dense representation ρ of $D(2,3,\infty)$ given by*

$$x \mapsto \begin{pmatrix} 0 & -1 & 2 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

Before we prove this theorem, we will describe how the matrices $\rho(x), \rho(y)$ can be obtained.

Definition 6. We let the map $\mathrm{Sym}^2 : M_{2\times 2} \to M_{3\times 3}$ be given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Though the map Sym^2 is defined on other matrices, we will not need the general definition here.

First, we set $\rho(y) := \mathrm{Sym}^2(y)$. To get $\rho(x)$, we compute $\mathrm{Sym}^2(x)$, and then conjugate by

$$M_t := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

where $t = 1$.

Remark 7. The Zariski denseness of the representation is important because it implies that the representation has an Anosov element. Showing Zariski denseness is useful because it does not rely on an ad hoc search through elements in a group to find an Anosov element. That said, there is a relatively simple matrix that is Anosov, namely

$$\rho(yx) = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 4 & -4 \\ 2 & -4 & 5 \end{pmatrix}$$

Also, Zariski dense sugroups of infinite index are called "thin" and constructing thin subgroups is of independent interest.

Proof. First, we justify why the representation ρ is faithful. While it would be clean to argue using the Hitchin component a la Long, Reid, and Thistlethwaite in [LRT11], we are instead using an explicit argument.

Remark 8. The construction of $\rho(x), \rho(y)$ is motivated by the notion of a Hitchin component, but this notion has not been defined for $D(2,3,\infty)$. The Hitchin component is defined for surface groups [Can23] and for fundamental groups of compact 2 dimensional orbifolds with negative orbifold Euler characteristic [Ale23]. This latter fact enables Long, Reid, and Thistlethwaite to verify that their representation in [LRT11] is faithful.

One may hope to define a Hitchin component for $D(2,3,\infty)$, but there seem to be some obstacles. If the usual notion of Hitchin component made sense for $D(2,3,\infty)$, then the representation ρ' given by $x \mapsto \mathrm{Sym}^2(x), y \mapsto M_{-1}\mathrm{Sym}^2(x)M_{-1}^{-1}$ would still be faithful. However, ρ' is not faithful since $\rho'((xy)^6) = id$.

We proceed with the argument.

To start, we attempt to use the ping-pong lemma to find a copy of F_2 (the free group on 2 generators) in $\operatorname{Im}\rho$. Here is the form of the ping-pong lemma we are using.

Lemma 2. *[CG] Let the group Γ act on the space X. Suppose we have elements $a, b \in \Gamma$ and disjoint subsets A^+, A^-, B^+, B^- in X such that*

$$\begin{aligned} a.(A^+ \cup B^- \cup B^+) &\subseteq A^+ \\ a^{-1}.(A^- \cup B^- \cup B^+) &\subseteq A^- \\ b.(B^+ \cup A^- \cup A^+) &\subseteq B^+ \\ b^{-1}.(B^- \cup A^- \cup A^+) &\subseteq B^- \end{aligned}$$

Then $\langle a, b\rangle = F_2$.

We have not yet explicitly verified this, but we suspect that regions A^+, A^-, B^+, B^- that are ellipse-shaped when seen in $\mathbb{RP}^2$ should work.

We have shown that ρ is injective up to finite index, so now we resolve the finite index issue. Denote the copy of F_2 by H. Since H is index 3 in $PSL(2,\mathbb{Z})$, it suffices to ensure that the elements of order 3 are not sent to identity by ρ. Recognizing $PSL(2,\mathbb{Z})$ as a triangle group $D(2,3,\infty)$, we see that all elements of order 3 are conjugate to y [Zie66]. But a group element has trivial image under a homomorphism if and only if another element conjugate to it does, and y has nontrivial image under ρ. Therefore, ρ is faithful, as desired.

Second, we justify why ρ is Zariski dense. By a theorem of Guichard [cite long, thist. 2k+1, Thm 3.1, also proved in Sam20 cited in Zshornack], if one can show that a given representation of a surface group leaves no form invariant, then its image is Zariski dense. We can check this property directly. A quadratic form in three variables can be written as a symmetric matrix

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Here e is a variable.

Let e_1, e_2, e_3 denote the 3 standard unit basis vectors in $\mathbb{R}^3$. If ρ leaves a form invariant, there must exist A so that

$$\begin{aligned}
e_1^T A e_2 &= (xe_1)^T A(xe_2) = (ye_1)^T A(ye_2) \\
e_1^T A e_3 &= (xe_1)^T A(xe_3) = (ye_1)^T A(ye_3) \\
e_2^T A e_3 &= (xe_2)^T A(xe_3) = (ye_2)^T A(ye_3) \\
e_1^T A e_2 &= (y^2 e_1)^T A(y^2 e_2)
\end{aligned}$$

This system of equations in the variables a, b, c, d, e, f has only the degenerate solution $a = b = c = d = e = f = 0$. So ρ cannot leave a form invariant, and is thus Zariski dense. □

6.3 Analog of Hurder's Theorem 7.22 for rep of $PSL(2, \mathbb{Z})$

This argument has been completed but not yet written up.

6.4 Surface Group Action on Nilmanifolds

In this subsection, we will (try to) construct a surface group action on a nilmanifold, that is not a torus. Let us denote $G = \pi_1(\Sigma_2)$. We will imitate the procedure described in [FKS11] on page 22. We summarize what we shall do here:

1. Describe the free 2-step nilpotent Lie algebra $\mathfrak{n}$ generated by X_1, X_2, X_3, Z_1, Z_2, and Z_3.
2. Create the free 2-step nilpotent Lie group N from $\mathfrak{n}$.
3. Define a lattice Γ of N.
4. Define the G-action on the elements X_1, X_2, X_3.
5. Induce the G-action in the following order:
   on elements Z_1, Z_2, Z_3, then on $\mathfrak{n}$, then on N, and finally on N/Γ.
6. Show that the G-action on N/Γ is Anosov.

6.4.1 Construction

Let $\mathfrak{n}$ be the free 2-step nilpotent Lie algebra of rank 3 over $\mathbb{R}$. For a more theory-heavy approach to these Lie algebras, we refer the reader to [PCD14] and [CR21]. From page 5 on [CR21], this Lie algebra can be represented in terms of 7-by-7 matrices in the following way

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & 0 & a & r & p & 0 \\ 0 & 0 & 0 & b & q & 0 & -p \\ 0 & 0 & 0 & c & 0 & -q & -r \\ 0 & 0 & 0 & 0 & c & b & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c, p, q, r \in \mathbb{R} \right\}, \tag{11}$$

with $\mathbb{R}$-vector space structure as one would expect and Lie brackets by $[X, Y] = XY - YX$. This is generated as a vector space by $X_1, X_2, X_3, Z_1, Z_2, Z_3$, which are matrices of the form (11) with zeros everywhere except an one at a, b, c, p, q, r, respectively. We remark that $\mathfrak{n}$ is not free as a Lie algebra. It can compute that $[\mathfrak{n}, \mathfrak{n}]$ consists of matrices $pZ_1 + qZ_2 + rZ_3$ for $p, q, r \in \mathbb{R}$. In turn, we can then see that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \{1\}$, so $\mathfrak{n}$ is indeed free and 2-step nilpotent. We note that its commutation relations are given by $[X_1, X_2] = Z_1, [X_2, X_3] = Z_2$, and $[X_1, X_3] = Z_3$ and zero otherwise.

The exponential map of a nilpotent Lie algebra is a diffeomorphism to its simply-connected Lie group, see Proposition 1.6.1 on [RF16]. Hence, we can apply the exponential map on $\mathfrak{n}$ to obtain its associated simply-connected Lie group N. As a matrix Lie algebra, the exponential map is given by (7). One can compute that for any $X \in \mathfrak{n}$, we have $X^i = 0$ for $i \geq 3$. Therefore, the simply-connected Lie group of $\mathfrak{n}$ is given by

$$\begin{aligned} N &= \left\{ I + X + \frac{1}{2}X^2 : X \in \mathfrak{n} \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 & 0 & a & \frac{ac}{2} + r & \frac{ab}{2} + p & \frac{a^2}{2} \\ 0 & 1 & 0 & b & \frac{bc}{2} + q & \frac{b^2}{2} & \frac{ab}{2} - p \\ 0 & 0 & 1 & c & \frac{c^2}{2} & \frac{bc}{2} - q & \frac{ac}{2} - r \\ 0 & 0 & 0 & 1 & c & b & a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c, p, q, r \in \mathbb{R} \right\} \end{aligned} \tag{12}$$

It can be verified that this is a 2-step nilpotent Lie group. Indeed, the center of N consists of matrices of the form (12) with $a = b = c = 0$ and $p, q, r \in \mathbb{R}$. Then it can be shown that the quotient $N/Z(N)$ is abelian computationally.

Let Γ be the set of matrices of the form (12) for $a, b, c \in \mathbb{Z}$ and $p, q, r \in \{n/2 : n \in \mathbb{Z}\}$. In Section 6.4.2 below, we show that it is a subgroup and, moreover, that it is a lattice.

Now let us define the $\pi_1(\Sigma_2)$-action on N/Γ. In Section 6.1, we proved that $G = \pi_1(\Sigma_2)$ has a faithful representation ρ into $\mathrm{SL}(3, \mathbb{Z})$. Hence, G has an action on $\mathbb{R}^3$ by matrix multiplication: $g \cdot v \mapsto \rho(g)v$ for $g \in G$ and $v \in \mathbb{R}^3$.

Intuitively, we want to identify the generators X_1, X_2, and X_3 of $\mathfrak{n}$ with the standard basis elements e_1, e_2, and e_3 of $\mathbb{R}^3$, respectively. Then, we can define the G-action on the X_is as it acts on the standard basis. More precisely, for $g \in G$, we can write $g \cdot e_i = \big(a_i, b_i, c_i\big)$ for some $a_i, b_i, c_i \in \mathbb{R}^3$. In turn, we define $g \cdot X_i \coloneqq a_i X_1 + b_i X_2 + c_i X_3$. We remark that $g \cdot e_i$ is just the i-th column of $\phi(g)$.

Next, we can naturally expand our action onto Z_i to preserve the Lie brackets by defining $g \cdot Z_1 \coloneqq (g \cdot X_1)(g \cdot X_2) - (g \cdot X_2)(g \cdot X_1)$ and similarly for Z_2 and Z_3. Finally, we can define the G-action on arbitrary elements of $\mathfrak{n}$ by doing so linearly. That is, for any $X \in \mathfrak{n}$, we can write uniquely $X = d_1 X_1 + d_2 X_2 + d_3 X_3 + d_4 Z_1 + d_5 Z_2 + d_6 Z_3$ as they form a basis for $\mathfrak{n}$. Then, we define $g \cdot X \coloneqq d_1(g \cdot X_1) + ... + d_6(g \cdot Z_3)$. This construction creates a well-defined G-action on the Lie algebra $\mathfrak{n}$, where elements act as Lie algebra automorphism.

We can now raise this to an action on the Lie group N. Recall, every element X' of N can be written as $X' = I + X + X^2/2$ for some $X \in \mathfrak{n}$. Then we can define $g \cdot X' \coloneqq I + g \cdot X + (g \cdot X)^2/2$. This preserves the exponential map and defines a G-action on N as a Lie group, where elements act as Lie group automorphisms.

Let us explicitly write out this action. For $g \in G$, can write

$$\rho(g) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

in $\mathrm{SL}(3, \mathbb{Z})$. Then for some arbitrary element X in N of the form (12), we have that $g \cdot X$ is

$$\begin{pmatrix} 1 & 0 & 0 & a_{12}b + a_{13}c + aa_{11} & R_1 & P_1 & \frac{1}{2}\left(a_{12}b + a_{13}c + aa_{11}\right)^2 \\ 0 & 1 & 0 & a_{22}b + a_{23}c + aa_{21} & Q_1 & \frac{1}{2}\left(a_{22}b + a_{23}c + aa_{21}\right)^2 & P_2 \\ 0 & 0 & 1 & a_{32}b + a_{33}c + aa_{31} & \frac{1}{2}\left(a_{32}b + a_{33}c + aa_{31}\right)^2 & Q_2 & R_2 \\ 0 & 0 & 0 & 1 & a_{32}b + a_{33}c + aa_{31} & a_{22}b + a_{23}c + aa_{21} & a_{12}b + a_{13}c + aa_{11} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} P_i &= \frac{1}{2}\left(a_{12}b + a_{13}c + aa_{11}\right)\left(a_{22}b + a_{23}c + aa_{21}\right) \\ &\quad + (-1)^{i-1}\left[\left(a_{11}a_{22} - a_{12}a_{21}\right)p + \left(a_{12}a_{23} - a_{13}a_{22}\right)q + \left(a_{11}a_{23} - a_{13}a_{21}\right)r\right] \\ Q_i &= \frac{1}{2}\left(a_{22}b + a_{23}c + aa_{21}\right)\left(a_{32}b + a_{33}c + aa_{31}\right) \\ &\quad + (-1)^{i-1}\left[\left(a_{21}a_{32} - a_{22}a_{31}\right)p + \left(a_{22}a_{33} - a_{23}a_{32}\right)q + \left(a_{21}a_{33} - a_{23}a_{31}\right)r\right] \\ R_i &= \frac{1}{2}\left(a_{12}b + a_{13}c + aa_{11}\right)\left(a_{32}b + a_{33}c + aa_{31}\right) \\ &\quad + (-1)^{i-1}\left[\left(a_{11}a_{32} - a_{12}a_{31}\right)p + \left(a_{12}a_{33} - a_{13}a_{32}\right)q + \left(a_{11}a_{33} - a_{13}a_{31}\right)r\right] \end{aligned}$$

As every coefficient a_{ij} is an integer, it is clear from above that our G-action on N preserves the lattice Γ; that is, for every $g \in G$ and $X \in \Gamma$, $g \cdot X \in \Gamma$. In turn, we have an induced G-action on the nilmanifold N/Γ by defining $g \cdot X/\Gamma \coloneqq (g \cdot X)/\Gamma$ for $X/\Gamma \in N/\Gamma$. This is well-defined as for any $XN = YN \in N/\Gamma$, we have $XY^{-1} \in \Gamma$. Since our action preserves Γ, we have $g \cdot XY^{-1} \in \Gamma$.

Each element act as a Lie group automorphism, so, in particular, are group homomorphisms. Hence, $(g \cdot X)(g \cdot Y)^{-1} = g \cdot (XY^{-1}) \in \Gamma$. Therefore, $(g \cdot X)\Gamma = (g \cdot Y)\Gamma$. Thus, $g \cdot (X\Gamma) = (g \cdot X)\Gamma = (g \cdot Y)\Gamma = g \cdot (Y\Gamma)$, so our G-action on N/Γ is well-defined, where each element acts as a diffeomorphism. We remark that we are not certain that this G-action on N/Γ is faithful. Let $g = g_0 g_1 g_3 \in G$. We give here a work-in-progress idea to show that g acts as an Anosov diffeomorphism on N/Γ. Its representation in $\mathrm{SL}_3(\mathbb{Z})$ with $t = 1$ is

$$\begin{pmatrix} 1072 & 529 & 314 \\ 264 & 131 & 77 \\ -441 & -218 & -129 \end{pmatrix}$$

As a linear map on $\mathbb{R}^3$, it has eigenvalues $\lambda_1 \approx 1073.21, \lambda_2 \approx 0.79, \lambda_3 \approx 0.00179$, with eigenvectors v_1, v_2, v_3. Moreover, it can be verified that no product of two eigenvalues has modulus 1. By our action defined above, it acts linearly in the Lie algebra $\mathfrak{n}$. It is straightforward to show that the action of g has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3$ with eigenvectors

$$\begin{aligned} V_1 &= v_{1,1}X_1 + v_{1,2}X_2 + v_{1,3}X_3, & U_1 &= V_1V_2 - V_2V_1, \\ V_2 &= v_{2,1}X_1 + v_{2,2}X_2 + v_{2,3}X_3, & U_2 &= V_2V_3 - V_3V_2, \text{ and} \\ V_3 &= v_{3,1}X_1 + v_{3,2}X_2 + v_{3,3}X_3, & U_3 &= V_1V_3 - V_3V_1, \end{aligned}$$

respectively. Again, note that the eigenvalues are all not modulus 1; in particular, $\lambda_1, \lambda_1\lambda_2, \lambda_1\lambda_3 > 1$ and $\lambda_2, \lambda_3, \lambda_2\lambda_3 < 1$. It can be computationally verified that V_1, V_2, V_3, U_1, U_2, and U_3 form a basis for $\mathfrak{n}$. Let us define $E_I^s = \mathrm{span}_{\mathbb{R}}\{V_2, V_3, U_2\}$ and $E_I^u = \mathrm{span}_{\mathbb{R}}\{V_1, U_1, U_3\}$. Intuitively, they form are the expanding and contracting subspaces of $\mathfrak{n}$. Note that $\mathfrak{n} = T_I N = E_I^s \oplus E_I^u$ and

1) $g(E_I^s) = E_I^s$ and $g(E_I^u) = E_I^u$.
2) There exist $C > 0$ and $\lambda > 1$ such that
    a) For all $n \geq 1$ and $v \in E_I^s$, we have $\|g^n(v)\| \leq (C/\lambda^n)\|v\|$.
    b) For all $n \geq 1$ and $v \in E_I^u$, we have $\|g^{-n}(v)\| \leq (C/\lambda^n)\|v\|$.

For each $A \in N$, define $E_A^s = d(R_A)_I(E_I^s)$ and $E_A^u = d(R_A)_I(E_I^u)$, where $R_A : N \to N$ is right-multiplication by A. Since R_A is a diffeomorphism of N, it follows that $d(R_A)_I : \mathfrak{n} = E_I^s \oplus E_I^u \to T_A N$ is a linear isomorphism, and thus $T_A N = E_A^s \oplus E_A^u$. We also define $F_{A\Gamma}^s = d\pi_A(E_A^s)$ and $F_{A\Gamma}^u = d\pi_A(E_A^u)$, where $\pi : N \twoheadrightarrow N/\Gamma$ is the canonical projection. The differential of π at any point is a linear isomorphism (since Γ is discrete), so we obtain $T_{A\Gamma}(N/\Gamma) = F_{A\Gamma}^s \oplus F_{A\Gamma}^u$.

To summarize, we have constructed a G-action on N/Γ, which we strongly believe, but have not proven, is faithful and Anosov. We have a candidate $g \in G$ that will act as an Anosov diffeomorphism on N/Γ and also candidates for the expanding and contracting subbundles of $T(N/\Gamma)$ corresponding to g.

6.4.2 Proof that Γ is a lattice

We show that Γ, consisting of matrices of the form (12) where $a, b, c \in \mathbb{Z}$ and $p, q, r \in \{n/2 : n \in \mathbb{Z}\}$, is a subgroup. Let X and Y be arbitrary elements of Γ in the forms below, where $p = p'/2, q = q'/2, r = r'/2, x = x'/2, y = y'/2, z = z'/2$ and $a, b, c, u, v, w, p', q', r', x', y', z' \in \mathbb{Z}$. We can see that the product of X and Y is of the form

$$
\begin{aligned}
XY &= \begin{pmatrix}
1 & 0 & 0 & a & \frac{ac}{2} + r & \frac{ab}{2} + p & \frac{a^2}{2} \\
0 & 1 & 0 & b & \frac{bc}{2} + q & \frac{b^2}{2} & \frac{ab}{2} - p \\
0 & 0 & 1 & c & \frac{c^2}{2} & \frac{bc}{2} - q & \frac{ac}{2} - r \\
0 & 0 & 0 & 1 & c & b & a \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & u & \frac{uw}{2} + z & \frac{uv}{2} + x & \frac{u^2}{2} \\
0 & 1 & 0 & v & \frac{vw}{2} + y & \frac{v^2}{2} & \frac{uv}{2} - x \\
0 & 0 & 1 & w & \frac{w^2}{2} & \frac{vw}{2} - y & \frac{uw}{2} - z \\
0 & 0 & 0 & 1 & w & v & u \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
&= \begin{pmatrix}
1 & 0 & 0 & a+u & \frac{ac}{2} + aw + r + \frac{uw}{2} + z & \frac{ab}{2} + av + p + \frac{uv}{2} + x & \frac{a^2}{2} + au + \frac{u^2}{2} \\
0 & 1 & 0 & b+v & \frac{bc}{2} + bw + q + \frac{vw}{2} + y & \frac{b^2}{2} + bv + \frac{v^2}{2} & \frac{ab}{2} + bu - p + \frac{uv}{2} - x \\
0 & 0 & 1 & c+w & \frac{c^2}{2} + cw + \frac{w^2}{2} & \frac{bc}{2} + cv - q + \frac{vw}{2} - y & \frac{ac}{2} + cu - r + \frac{uw}{2} - z \\
0 & 0 & 0 & 1 & c+w & b+v & a+u \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
&= \begin{pmatrix}
a+u & \frac{1}{2}(a+u)(c+w) + \frac{aw}{2} - \frac{cu}{2} + r + z & \frac{1}{2}(a+u)(b+v) + \frac{1}{2}(av - bu) + p + x & \frac{1}{2}(a+u)^2 \\
b+v & \frac{1}{2}(b+v)(c+w) + \frac{1}{2}(bw - cv) + q + y & \frac{1}{2}(b+v)^2 & \frac{1}{2}(a+u)(b+v) + \frac{1}{2}(bu - av) - p - x \\
c+w & \frac{1}{2}(c+w)^2 & \frac{1}{2}(b+v)(c+w) + \frac{1}{2}(cv - bw) - q - y & \frac{1}{2}(a+u)(c+w) + \frac{1}{2}(cu - aw) - r - z \\
1 & c+w & b+v & a+u \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \\
&= \begin{pmatrix}
1 & 0 & 0 & a+u & \frac{(a+u)(c+w)}{2} + \frac{aw-cu+r'+z'}{2} & \frac{(a+u)(b+v)}{2} + \frac{av-bu+p'+x'}{2} & \frac{1}{2}(a+u)^2 \\
0 & 1 & 0 & b+v & \frac{(b+v)(c+w)}{2} + \frac{bw-cv+q'+y'}{2} & \frac{1}{2}(b+v)^2 & \frac{(a+u)(b+v)}{2} - \frac{av-bu+p'+x'}{2} \\
0 & 0 & 1 & c+w & \frac{1}{2}(c+w)^2 & \frac{(b+v)(c+w)}{2} - \frac{bw-cv+q'+y'}{2} & \frac{(a+u)(c+w)}{2} - \frac{aw-cu+r'+z'}{2} \\
0 & 0 & 0 & 1 & c+w & b+v & a+u \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\end{aligned}
$$

and the inverse of X is

$$
\begin{pmatrix}
1 & 0 & 0 & -a & \frac{ac}{2} - r & \frac{ab}{2} - p & \frac{a^2}{2} \\
0 & 1 & 0 & -b & \frac{bc}{2} - q & \frac{b^2}{2} & \frac{ab}{2} + p \\
0 & 0 & 1 & -c & \frac{c^2}{2} & \frac{bc}{2} + q & \frac{ac}{2} + r \\
0 & 0 & 0 & 1 & -c & -b & -a \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

which are both in the form of elements of Γ. Hence, Γ is a subgroup of the free 2-step nilpotent Lie group N.

Let us now prove that Γ is a lattice. We will apply Theorem 2.12 in [Mad72]. Observe that the basis X_i, Z_i for the Lie algebra $\mathfrak{n}$ has structural constants which are rational. Let $\mathfrak{n}_0$ be the $\mathbb{Q}$-vector space spanned by X_i, Z_i. Consider the set Γ_0 of matrices in $\mathfrak{n}$ of the form (11) with $a, b, c \in \mathbb{Z}$ and $p, q, r \in (1/2)\mathbb{Z}$. It is clear that $\mathbb{Z}^3 \times ((1/2)\mathbb{Z})^3$ is a lattice in $\mathbb{R}^6$, so interpreting $\mathfrak{n}$ as $\mathbb{R}^6$, we can see that Γ_0 is a lattice in $\mathfrak{n}$. Observe that Γ_0 is contained in $\mathfrak{n}_0$. The rank of the lattice Γ_0 is the dimension of the $\mathbb{R}$-span by Γ_0, which is certainly the same as the dimension of $\mathfrak{n}$, so it is of maximal rank. We see that $\exp(\Gamma_0) = \Gamma$, and we proved earlier that it is a subgroup of N. Hence, by the discussion below Theorem 2.12 of [Mad72], Γ is a lattice in N.

7 Future directions

One of our future directions is determining structural stability for our $\pi_1(\Sigma_2)$-action on $\mathbb{T}^3$. To do this, we would take the $D(3,3,4)$ action on $\mathbb{T}^3$ given to us by [LRT11] and show it is not topologically rigid. Then, we would prove that finite index subgroups of non topologically rigid groups being rigid. This result seems likely to be true given some empirical evidence, but we would want to check it rigorously. Since $\pi_1(\Sigma_2)$ is finite index in $D(3,3,4)$, as we showed earlier in the writeup, we would obtain as a corollary that the $\pi_1(\Sigma_2)$-action on $\mathbb{T}^3$ is not topologically rigid. Since $\pi_1(\Sigma_n)$ is finite index in $\pi_1(\Sigma_2)$, we would also get as a corollary that the $\pi_1(\Sigma_n)$-action on $\mathbb{T}^3$ is not topologically rigid.

Secondly, we hope to obtain a $\mathrm{PSL}(2,\mathbb{Z})$ action on $\mathbb{T}^3$ with Anosov element. We think we know how to do this, but need to iron out some details. After showing faithfulness of the representation $\mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{SL}(3,\mathbb{Z})$, we would then argue, in a generalization of Hurder's argument, that the induced action on $\mathbb{T}^3$ fails to be topologically deformation rigid.

Thirdly, we hope to construct an Anosov $\pi_1(\Sigma_2)$-action on the quotient of the free 2-step nilpotent Lie group N on three generators by a lattice $\Gamma \subseteq N$. Most of it seems to work, but we are still verifying (1) the faithfulness of the action on the nilmanifold level, and (2) the existence of an Anosov element. We believe that both of these verifications should come out positive, but we're still in the process of working out the details. If the arguments go through, we may try and consider Anosov $\pi_1(\Sigma_2)$-actions on free k-step nilmanifolds on n generators more generally, or other more broad families of nilmanifolds.

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